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Disadvantageous Syndicates and Stable Cartels: The Case of the Nucleolus*

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This paper considers a bilateral market with two complementary commodities and gives a rationale for Aumann's paradox. The relationship between the notion of strong stability of a syndicate, i.e., the property that no group of players wants to exit or to enter the syndicate, and the notion of disadvantageous syndicates is summarized in two results. If the two sides of the market are balanced in terms of endowments, every syndicate is strongly stable. If the two sides of the market are not balanced in terms of endowments, then being advantageous, in Aumann's definition, is necessary and sufficient for a syndicate to be strongly stable. *Journal of Economic Literature.* Classification Numbers: 022, 611. © 1987 Academic Press, Inc.

1. INTRODUCTION

A monopoly situation is usually understood as one in which one enterprise succeeds in producing a large proportion, or all, of a given commodity. One generally expects that the monopoly sets the price and earns an extra profit with respect to the competitive situation. So if we consider several agents in an exchange economy and if we ask them whether they would like to collude and, in doing so, to obtain monopoly power, or if they would prefer to play the symmetric game—i.e., to stay in the competitive framework—it is tempting to guess that the first alternative would be chosen.

In that respect, Aumann's paper "Disadvantageous Monopolies" [4] is disturbing. Indeed, Aumann shows that in a mixed market (see Shitovitz [32]), it may be the case that the players in the atom always receive less

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within the core than with the competitive solution. He concludes that the core fails to reflect the monopolist's power, and consequently, that it is not the proper vehicle for explaining the advantage of the monopolist. He ends by proposing the Shapley value as a more significant concept, but later papers (Gardner [10], Guesnerie [12]) prove that the Shapley value has the same undesirable property as the core. Moreover, recent papers, such as Okuno *et al.* [22], Salant *et al.* [27], Szidarowsky and Yakowitz [36], show that the non-cooperative solutions are not better candidates than the Shapley value and the core. One may wonder at this point about the relevance of any game theoretical explanation of monopoly power. We argue in this paper that there is no paradox in Aumann's result, and more important, no contradiction with economic intuition.

Indeed, an atom in a mixed market is usually understood as a consequence of binding agreements between different agents, what game theory calls a syndicate (Dreze and Gabszewitz [7]). The question which must be asked is in fact the one with which we started: do people want to share monopoly power and do they gain in doing so? Here the industrial organization literature does not provide a clear answer. Indeed, some emperical studies show either a negative relationship between the degree of collusion and the profitability of enterprises (Asch and Seneca [1]) or a nonsignificant variation in the profitability after collusive behavior (Scherer [28, pp. 138, 242, 544f.]).

Curiously, Aumann himself gives an explanation of his paradox when he states, "Monopoly power is based on what the monopolist can prevent other coalitions from getting. His strength lies in his threat possibilities, in the bargaining power engendered by the harm he can cause by refusing to trade." (Aumann [4, p. 10]). The same sort of idea is expressed by Kats: "monopolistic power is revealed not only through [the monopolist's] ability to set prices but also through his power to determine the strategies available to the other agents." (Kats [15, p. 253]). If we look at a two-sector economy, explaining the disadvantage of a syndicate in one sector is then the same as explaining why every marginal player in the other sector gains more power to threaten than in the competitive situation.

The present paper develops this idea and gives a rationale for the appearance of disadvantageous syndicates in a two-sided market with two complementary goods. It turns out that the disadvantageousness of a syndicate depends upon the relative rarity of the commodity that its members own. Stated differently, incentives for collusive behavior decrease with the rarity of the commodity. Consequently, the ratio of the available quantities of the two commodities becomes a proxy for measuring monopoly power. To obtain this result, we make use of a cooperative game concept, the nucleolus. We remark that the same result can be obtained with the Shapley value (see Legros [16]). We interpret the condition for a syndicate to be advantageous as a necessary condition for its appearance. In addition, because it is well known that collusive behavior is not in general a stable process, it is of interest to pose the problem of the stability of a syndicate. We show that the above condition is also sufficient for a syndicate to be stable with respect to the exit or the entrance of any group of players. We finally interpret this result by means of an endogenous process of coalition formation.

2. The Model

The economy consists of two types of players and of two commodities A and B. Let P and Q be the sets of players of the first and of the second types respectively, and suppose that each player of type $P(Q)$ has an initial endowment of one unit of commodity A (B). Each player i has the same utility function $u_i(a, b) = \min(a, \lambda b)$, where (a, b) represents the bundle of goods A and B, and $\lambda > 0$ is a given parameter. If money exists in the economy and if side payments are allowed, this economy generates a market game whose characteristic function is,[1]

$$v(S) = \min(|S \cap P|, \lambda \cdot |S \cap Q|) \qquad \text{for all} \quad S \subset P \cup Q. \tag{1}$$

An interpretation of this game for $|P| = 2$, $|Q| = 3$, and $\lambda = \frac{1}{2}$ is found in Maschler [19, p. 186] (see also Postlewaite and Rosenthal [25]):

> Each of two manufacturers owns two machines that can be operated only by skilled workers. There are exactly three available skilled workers, each willing to work 8 hr a day. When a worker operates the machine during 8 hr, he produces an item that can be sold at a net profit of 1/2 unit of money.

The problem is then to divide the total net profit $v(P \cup Q)$ between workers and owners of machines. For a game in characteristic form, different solution concepts can be used, each of them corresponding to certain behavior of the players or to a certain normative criterion. We shall suppose in the following that the allocation scheme is given by the nucleolus, a concept introduced in Schmeidler [29]. Note that this choice is not crucial here since most of the results hold when the Shapley value is used (Legros [16]).

[1] This is a generalization of the well-known gloves game (see Shapley and Shubik [31], for instance). The use of a side-payment market game may seem restrictive. Yet, note that Guesnerie [12, pp. 240–241] proves that for homogeneous markets, the weights associated with every Shapley value allocation (Shapley [30]) are $(1/n, \ldots, 1/n)$, where n is the number of players, i.e., that the Shapley value levels of utility are in fact the Shapley value assignments of the corresponding game with transferable utilities.

The nucleolus of a side-payment game (N, v) is the imputation which minimizes the greatest dissatisfaction of any coalition in the following sense.[2] Let A_v be the set of admissible coalitions for the game (N, v) and X the set of imputations, i.e., of individually rational and Pareto optimal allocations,

$$X = \left\{ x \in \mathbb{R}^n \,\middle|\, x_i \geq v(\{i\}),\ i = 1,..., n \text{ and } \sum_{i \in N} x_i = v(N) \right\}. \tag{2}$$

The excess $e(S, x) = v(S) - \sum_{i \in S} x_i$ of a coalition $S \in A_v$ with respect to the imputation x is a measure of the dissatisfaction (positive or negative) that coalition S feels when x is proposed. If we consider the vector $\Theta(x)$ in $\mathbb{R}^{|A_v| - 1}$ of excesses ranked in decreasing order, the nucleolus is the vector x such that $\Theta(x)$ is inferior to every other vector $\Theta(y)$, $y \neq x$, in the lexicographical ordering on $\mathbb{R}^{|A_v| - 1}$.[3] Thus the nucleolus implements in a certain respect a notion of justice à la Rawls [26]. Existence, uniqueness, and continuity of the nucleolus are proved in Schmeidler [29]. Below, the nucleolus of the game v is denoted by nu^v.

3. The Advantageous Syndicates

Suppose that a syndicate is formed on one side of the market, while the other side stays unorganized. Let SP and SQ denote respectively the partitions $\{P, \langle Q \rangle\}$ and $\{\langle P \rangle, Q\}$ where $\langle T \rangle$ stands for $\{\{i_1\},..., \{i_t\}\}$, $i_1,..., i_t \in T$. Each partition SP or SQ generates a game with $|Q| + 1$ or $|P| + 1$ players in which the syndicate P or Q is considered as a single player. These games, denoted by v^{SP} and v^{SQ}, are obtained from v by restricting the set of coalitions to those which either contain the syndicate or exclude it.

Aumann's notion of advantage is the following. A syndicate I $(I = P, Q)$ is *advantageous* if all its members gain more, and at least one member gains strictly more, in the syndicate game than in the competitive game. Because the nucleolus treats symmetric players equally (this follows a result of Maschler and Peleg [20]) and with the mild assumption that the gains of the syndicate are equally shared between its members, the above definition can be formally written as

$$I\ (= P, Q) \text{ is advantageous} \Leftrightarrow nu_i^v < nu_I^{v^I}/|I| \qquad \text{all} \quad i \in I. \tag{3}$$

[2] For some economic applications of this concept, see Chetty *et al.* [5], Littlechild [18], Shubik and Young [33].

[3] I.e., $\Theta(x) <_{\text{lex}} \Theta(y) \Leftrightarrow$ there exist $i_0 \in \{1,..., 2^n - 1\}$ s.t. $\Theta_i(x) = \Theta_i(y)$, if $i < i_0$ and $\Theta_{i_0}(x) < \Theta_{i_0}(y)$.

In a companion paper (Legros [17]), the author presents the following algorithm to compute the nucleolus of the game v. Denote by $[x]$ the greatest integer smaller than x. Consider first the case $\lambda \leqslant p/q$ and define $k=[q/p]$, $\underline{\mu}=q/p$, $\bar{\mu}=(q-k)/(p-1)$, where $p=|P|$ and $q=|Q|$. Find t^* such that $t^*=\min\{t \mid t \in \operatorname{argmax}_{t \in M} t/[\bar{\mu}t]\}$, where $M=\{t \in \{1,..., p-1\} \mid [\bar{\mu}t]=[\underline{\mu}t]+1\}$. It can be proved that t^* exists. Then, if $\lambda \leqslant t^*/[\bar{\mu}t^*]$, $nu_i^v=0$ if $i \in P$ and $nu_j^v=\lambda$ if $j \in Q$, and if $\lambda \in [t^*/[\bar{\mu}t^*], p/q]$, $nu_i^v=(q/2)((\lambda[\bar{\mu}t^*]-t^*)/(p[\bar{\mu}t^*]-qt^*))$ if $i \in P$ and $nu_j^v=\frac{1}{2}(\lambda+(p-\lambda q)\, t^*/(p[\bar{\mu}t^*]-qt^*))$ if $j \in Q$. The same sort of algorithm applies when $\lambda \geqslant p/q$. Precisely, if $\tilde{t}$ is defined by by,[4] $\tilde{t}=\min\{t \mid t \in \operatorname{argmin} t/[\bar{\mu}t]\}$, where $\bar{\mu}=q/p$; then if $\lambda \geqslant \tilde{t}/[\bar{\mu}\tilde{t}]$, $nu_i^v=1$ if $i \in P$, and $nu_i^v=0$ if $j \in Q$, and if $\lambda \in [p/q, \tilde{t}/[\bar{\mu}\tilde{t}]]$, the nucleolus is

$$nu_i^v = 1 - \frac{q}{2}\frac{\tilde{t}-\lambda[\bar{\mu}\tilde{t}]}{q\tilde{t}-p[\bar{\mu}\tilde{t}]} \qquad \text{if } i \in P$$

and

$$nu_j^v = \frac{p}{2}\frac{\tilde{t}-\lambda[\bar{\mu}\tilde{t}]}{q\tilde{t}-p[\bar{\mu}\tilde{t}]} \qquad \text{if } \ j \in Q.$$

For the syndicate games v^{SP} and v^{SQ}, there exists a quite simple characterization of the nucleolus. Indeed, we show in Appendix 1 that the unorganized players always get half of their incremental contributions, i.e., that $nu_i^{v^I}=\frac{1}{2}[v(P \cup Q)-v(P \cup Q \backslash \{i\})]$ for all $i \notin I$. In other words, when players of a given type form a syndicate, they are able to appropriate half of the incremental contrbutions of all the other players.[5] This can be interpreted as a consequence of monopoly power. The question is then to know if this monopoly power—commonly shared by the members of the syndicate—is not in fact weaker in the game v^I than in the game v.

It is possible to draw the gaphs of $nu_i^{v^I}$ and of $\sum_{i \in I} nu_i^v$ $(I=P, Q)$ with respect to λ. This is done in Figs. 1 and 2.

Observe that the parameter λ is a measure of the relative rarity of the two commodities. If $\lambda < p/q$ for instance, the quantity $(p-\lambda q)$ of commodity A can be thrown away *without changing* the total profit of λq; in this respect, commodity A is abundant in the economy. It turns out that λ is also a good proxy for measuring monopoly power. Indeed, a direct consequence of Figs. 1 and 2 is the following result.

[4] Note that argmin is determined with respect to *all* t here.

[5] In Chetty *et al.* [5], the same characterization is obtained.

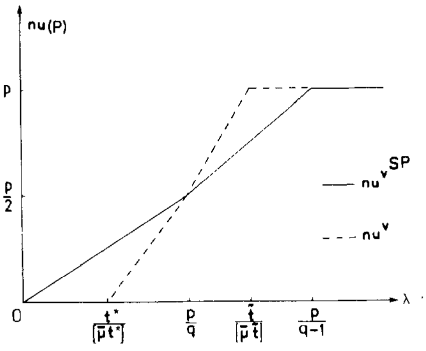

FIGURE 1

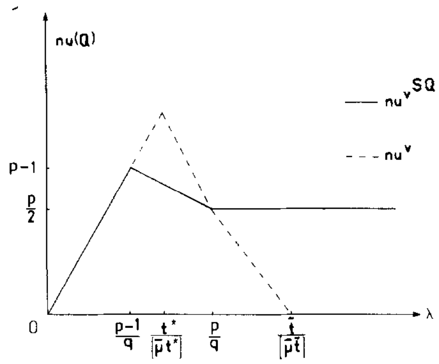

FIGURE 2

PROPOSITION 1. *A syndicate is advantageous if and only if its members initially own the abundant commodity,*

$$(P \text{ advantageous}) \Leftrightarrow (\lambda < p/q)$$

$$(Q \text{ advantageous}) \Leftrightarrow (\lambda > p/q).$$

Intuition suggests that owners of machines are in a weaker position than workers when the quantity of machines is relatively abundant in the economy (and conversely). Following this idea, when $\lambda = p/q$, we expect that neither side of the market will have greater power than the other. The nucleolus clearly takes account of these two insights. On one hand, the total payoffs of players on both sides are equal when $\lambda = p/q$ and, on the other hand, the sum of the payoffs of players on one side is less than the sum of the payoffs of players on the other side whenever the former players own the abundant commodity. We interpret this result as a consequence of the monopoly power of the players owning the rare commodity, i.e., their ability to announce credible threats (for instance, a worker can threaten a manufacturer not to work on his machine, and this threat is credible since machines are relatively abundant).

Proposition 1 states that the players who own the abundant commodity can gain by forming a syndicate. The reason here is that the syndicate which owns the abundant commodity can now make a credible threat (if it refuses to trade there is no profit available). The reasoning is at the margin: in the competitive situation a marginal worker can threaten a marginal manufacturer (if $\lambda < p/q$), whereas in the syndicate game, a marginal worker must deal with all the manufacturers as a whole. Observe finally that when the players owning the abundant commodity collude, they gain as much as if the two sides of the market were balanced in terms of endowments. Indeed, if for instance $\lambda < p/q$, $nu_P^{v^{SP}} = \lambda q/2$, and $\lambda q/2$ is the sum of the nucleolus payoffs of the players of type P only if $\lambda = p/q$.

Now, by reversing the previous argument, we are able to understand why a syndicate is disadvantageous—i.e., why players in the syndicate do not obtain more than in the competitive situation. If $\lambda > p/q$ for instance, each player of type P has a natural advantage over players of type Q; in fact, at the margin, each player of type P has "monopoly power" over the players of type Q who are not hired by the $|P|-1$ other players of type P. Proposition 1 says that by colluding, players of type P can lose part of this natural power, mainly because they are no longer able to individualize their threats.

We finally note that the players owning the rare commodity do not lower their payoffs by forming a syndicate only if λ is such that this commodity is still rare when a player owning the abundant commodity exits the market, i.e., when $\lambda(q-1) > p$ for the syndicate P and $\lambda q < p-1$ for the syndicate Q. This observation sustains our previous analysis since in this situation the syndicate can individualize its threats: every player owning the abundant commodity knows that the syndicate can obtain the maximum profit without his/her cooperation. As a matter of comparison, the author shows in Legros [16] that if the Shapley value is used, the players owning the rare commodity do not lose by colluding only if they can reach the maximum profit with the cooperation of only one player of the other side—i.e., when $\lambda > p$ for the syndicate P or when $\lambda q < 1$ for the syndicate Q. We can say that the nucleolus defines monopoly power at the margin, while the Shapley value does it on the whole.

4. The Stability of a Syndicate

The notion of stability which is induced by Aumann's definition of advantageousness is clearly restrictive since it compares two extreme situations (existence versus nonexistence of a syndicate). In Dreze and Gabszewitcz [7] and in Guesnerie [12], an alterative definition of stability is given: if all the players of a given type are not members of a given syndicate, this syndicate is stable if no syndicated player prefers the situation of an unsyndicated player and conversely.[6] This definition is attractive but may not be pertinent. Indeed, suppose that a syndicated agent envies the situation of an unsyndicated player, i.e., suppose that the syndicate does not satisfy the above stability condition. A natural consequence of this must be that this player wants to leave the syndicate. But in doing so, he changes the structure of the economy: the number of players increases and, moreover, the possibilities of some coalitions change. A rational, well-

[6] This is the marginal stability in Dreze and Gabszewitcz [7] and the B-stability in Guesnerie [12].

behaving agent must know this, and even if he envies another player, he may not want to leave the syndicate, because his payoff can be, in fact, smaller in the new market game.

A definition of stability must then take account of the influence of every player's decision on the organization of the economy. In a first approach we can define as stable a syndicate for which no player wants to change her status (i.e., syndicated–unsyndicated or from one syndicate to another) because—supposing that the other players maintain the same status—her allocation cannot be more in the resulting market game. Papers by d'Aspremont *et al.* [2], Donsimoni *et al.* [6] follow this approach. This definition, however, supposes that the other players behave in a Nash fashion, i.e., that each player expects that the other players will not change their status when he does change. This conjecture is central to all the literature on cartels (see Stigler [35], Osborne and Pitchik [24], Green and Porter [11]); and, indeed, preventing a member from cheating, or exiting, is vital for a cartel. Yet, if we can suspect a single player of conspiring against the cartel, there is no reason for not suspecting a group of players of conspiring. Indeed, if we agree with Smith [34, p.232] that "People of the same trade seldom meet together, even for merriment or diversion, but the conversation ends in a conspiration against the public," we can believe that he who lives by conspiracy will die by conspiracy.

What we need, therefore, is a Nash concept for coalitions, i.e., that for a syndicate to be stable, no group of players wants to change its status (always knowing that if anybody changes, the resulting market game is also changed, and so the final allocation). This is clearly the strongest definition of stability that can be given, and it is worthwhile to know if any syndicate can be stable according to this definition. We give an answer to this question here for the market game defined by (1).

Formally,[7] suppose that any syndicate, irrespective of its size, can form, the only restriction being that a syndicate must be constituted of players of the same type. So, if Π_P and Π_Q, are respectively, the class of partitions of P and Q, the only admissible partitions of $P \cup Q$ are elements of the set Π:

$$\Pi = \{\pi_P \cup \pi_Q \mid \pi_P \in \Pi_P, \pi_Q \in \Pi_Q\}.$$

If $\pi \in \Pi$, then v^π is the $|\pi|$-player game derived from v by restricting v to coalitions that are unions of members of π. For a syndicate S of a given partition $\pi \in \Pi$, $nu_S^{v^\pi}$ denotes the nucleolus payoff of S in v^π (note that $\sum_{S \in \pi} nu_S^{v^\pi} = v^\pi(P \cup Q) = v(P \cup Q)$). As before, we suppose that a syndicate shares its gains equally among its members. For ease of exposition, define

[7] I would like to thank the referee for his/her suggestions which greatly improved the readability of the following part of this section.

for $i \in S \in \pi$, $nu_i^{v^\pi} = nu_S^{v^\pi}/|S|$ (remember that i is not a player in v^π, so $nu_i^{v^\pi}$ is not actually the nucleolus payoff of i in v^π, except when $S = \{i\}$).

Given a partition $\pi \in \Pi$, it is of interest to know if any group of players (a group can contain players of either type) is better off with another partition $\tilde{\pi} \in \Pi$. A simple definition of stability for a partition is the following.

DEFINITION 1. A partition $\pi = \pi_P \cup \pi_Q \in \Pi$ is *stable* if there is no $T \subset P \cup Q$ and no $\tilde{\pi} = \tilde{\pi}_P \cup \tilde{\pi}_Q \in \Pi$ such that

(1) $nu_i^{v^\pi} \leqslant nu_i^{v^{\tilde{\pi}}}$, all $i \in T$,

(2) $nu_i^{v^\pi} < nu_i^{v^{\tilde{\pi}}}$, for some $i \in T$.

Lines (1) and (2) state that every mover is better off and at least one is strictly better off in the new partition.

Note that this definition implies a notion of stability for a syndicate: a syndicate $S \in \pi$ is stable if no group of players (inside and/or outside S) wants to move from π to another partition $\tilde{\pi}$. In fact, it is clear that in a stable partition, every syndicate is stable by this definition.

However, when we analyse the stability of a syndicate, there is no reason to start with a given organization of the economy: if a syndicate is stable, it should be so irrespective of the organization of the players outside the syndicate. The following definition expresses this idea.

DEFINITION 2. A syndicate I is *strongly stable* if every coalition structure that contains I is a stable partition.

The following proposition is proved in Appendix 2 and characterizes the strongly stable syndicates in a simple way.

PROPOSITION 2. (1) *Let $I \in \{P, Q\}$ and $\lambda \neq p/q$. I is advantageous if and only if I is strongly stable.*

(2) *If $\lambda \neq p/q$ and I is a strongly stable syndicate, then $I \in \{P, Q\}$.*

(3) *If $\lambda = p/q$ every syndicate is strongly stable.*

The following is immediate.

COROLLARY. *If $I \in \pi \in \Pi$, $I \in \{P, Q\}$, and I is advantageous then π is a stable partition.*

At this point, it is interesting to note that one can easily design a mechanism together with an equilibrium concept which leads to stable structures. The following mechanism shares many of the features introduced in Hart and Kurz [14]. Let A be the set of possible coalitions and $A_i \subset A$ the set of coalitions in which i is a member. For each player i

define a strategy to be an announcement of a coalition $S_i \in A_i$ to which she would like to belong, and suppose that $S_i \subset I$ if $i \in I \in \{P, Q\}$. Suppose now that there exists a certain mapping Ψ which assigns to each $p+q$-tuple of strategies $(S_1, ..., S_{p+q})$ in the set $A_1 \times \cdots \times A_{p+q}$ a given partition B of the players, i.e., $B = \Psi(S_1, ..., S_{p+q})$. The models γ and δ of Hart and Kurz [14] are examples of possible Ψ.[8] Finally, a market game v^B corresponds to the partition B by restricting the game v to coalitions generated by B. Formally, we have $v^B(S) = v(S)$ if $(i \in S \cap B_j) \Rightarrow (k \in S$, for all $k \in B_j)$ and $v^B(S) = 0$ otherwise.

Suppose that the players know that the total gain of the exchange is allocated with respect to the nucleolus. Then, we can define the normal form game $\Gamma = (S_i, nu_i^{v^B}, i \in P \cup Q)$ where $S_i \in A_i$, $B = \Psi(S_1, ..., S_{p+q})$, and $nu_i^{v^B}$ is defined as before. Different equilibrium concepts can be defined for this game Γ. For instance, the Nash equilibrium is a $p+q$-tuple $(\tilde{S}_1, ..., \tilde{S}_{p+q})$ such that for all $i \in P \cup Q$, $nu_i^{v^B} \geqslant nu_i^{v^{B-i}}$ where $B = \Psi(\tilde{S}_1, ..., \tilde{S}_{p+q})$ and $B_{-i} = \Psi(\tilde{S}_1, ..., \tilde{S}_{i-1}, S_i, \tilde{S}_{i+1}, ..., \tilde{S}_{p+q})$, $S_i \neq \tilde{S}_i$, i.e., such that no player has an incentive to leave his syndicate or to enter a new syndicate. Clearly, the possibility for the player $i \in B_j \in B$ to enter another syndicate $B_k \in B_{-i}$ if he changes his strategy depends upon the process Ψ which is adopted. When we allow coalitions to change strategies, we generalize in a certain way the Nash concept and we obtain the strong equilibrium of Aumann [3].[9] A strong equilibrium is a $p+q$-tuple of strategies $(\tilde{S}_1, ..., \tilde{S}_{p+q})$ such that for all coalitions T, the following does not hold,

$$nu_i^{v^{B-T}} \geqslant nu_i^{v^B} \qquad \text{for all} \quad i \in T,$$

and

$$nu_j^{v^{B-T}} > nu_j^{v^B} \qquad \text{for one} \quad j \in T.$$

where $B_{-T} = \Psi(S_1, ..., S_{p+q})$ with $S_i = \tilde{S}_i$ if $i \in P \cup Q \backslash T$ and $S_i \neq \tilde{S}_i$ if $i \in T$.

By Proposition 2, we can deduce that the set of strong equilibria of the game Γ is nonempty. Precisely, it consists of the $p+q$-tuples of strategies $(S_1, ..., S_{p+q})$ such that $\Psi(S_1, ..., S_{p+q}) = (I, I^c)$ where I is the advantageous syndicate, and I^c is *any* partition of the complementary set of players. This result is true whenever such a partition belongs to $\Psi(A_1 \times \cdots \times A_{p+q})$. For instance, both models γ and δ of Hart and Kurz [14] satisfy this condition, and if we consider these models, we can say that it is a strong equilibrium

[8] In the model γ a coalition B_j is an element of the final partition if all players in B_j have the same strategy B_j. In the model δ a coalition B_j belongs to the final partition if all the players in B_j announce the same strategy $S = S_i$ with $B_j \subset S$.

[9] For applications of this concept, see Dubey [8], Dubey and Shubik [9], Young [37].

strategy for players with the abundant commodity to announce $S_i = P$ or $= Q$, whatever the strategies of the other players on the other side are.

5. Concluding Remarks

In this model, the players who are on the "advantageous" side of the market will always collude; the players on the other side are then indifferent to their own organization. So, the bilateral monopoly is only a special case in this model. Note that in the model of Hart [13], the bilateral monopoly is the *only* stable organization.[10]

A relevant question would be to know if there exist situations in which some players lose when other players join their syndicates. This is another important aspect of the "entering" process into a syndicate: both incomers and former members should gain or at least not lose. Even if this requirement does not appear in our analysis, it is clear that all the players in a syndicate gain when the rest of the players of their own type join (if they initially own the abundant commodity). Since no syndicate of players with the rare commodity is strongly stable, the question is irrelevant for these players.

What do players in T do when they leave their respective syndicates? Also what happens to the rest of the players? These questions are of importance and it might be necessary in more general models to assert a particular behavior of the players, i.e., which changes in the original coalition structure are acceptable. In Definition 1, we only ask for a syndicate to consist of players of the same type. However, more particular restrictions can be imagined. For instance, we can suppose that when a group of players "moves," each player in this group can either join what is left from a syndicate of the original coalition structure, or can form with other "movers" a new syndicate (with the restriction that only players of the same type can collude.) Formally, we can impose the two following conditions:

(i) $A \in \tilde{\pi}_P \Rightarrow A = A_1 \cup A_2$ for some $A_1 \in \{E \backslash T \mid E \in \pi_P\} \cup \{\varnothing\}$ and for some $A_2 \subset T \cap P$.

(ii) $B \in \tilde{\pi}_Q \Rightarrow B = B_1 \cup B_2$ for some $B_1 \in \{F \backslash T \mid F \in \pi_Q\} \cup \{\varnothing\}$ and for some $B_2 \subset T \cap Q$.

[10] The markets considered in Hart [13] have a finite number of types of agents and a continuum of agents of each type. The sets of goods owned by all of the types are disjoint one from the other, and the von Neumann and Morgenstern solution is used. For another use of the von Neumann–Morgenstern solution to the problem of cartel stability, see Morgenstern and Schwödiauer [21].

Finally, one wonders if most of the results stem from the particular utility function and it this analysis can be generalized. Note first that a key feature of the analysis is the definition of rarity for a commodity which necessitates some complementarity between the two goods. This is clearly restrictive. For instance, with a utility function $u_i(a, b) = (ab)^{1/2}$, yielding a total profit of $(pq)^{1/2}$, it is possible to argue that if $p < q$, commodity B is abundant and commodity A is rare. But, if some quantities of commodity B are thrown away the total profit will decrease. Consequently, the threat possibilities are not directly related to the rarity of the commodity owned by a player. This does not mean that another analysis could not be performed with more general utility functions. In fact, Gardner [10] presents, within a different model, a nice characterization of disadvantageous syndicates for all homogeneous utility functions. However, for general utility functions, homogenous or not, the prospect of obtaining a simple characterization of the strongly stable syndicates seems unlikely.

There is nevertheless a simple way to introduce some substitution between the two commodities in our model. Consider the utility function $u_i(a, b) = \min(a, \alpha a + \lambda(1-\alpha)\, b)$ where $\alpha \in [0, 1]$, implying the characteristic function $v_\alpha(S) = \min(|S \cap P|, \alpha|S \cap P| + \lambda(1-\alpha)|S \cap Q|)$. Then, when $\lambda > p/q$, commodity B is abundant in our definition, but when $\lambda < p/q$, commodity A is not abundant in the strict sense of our definition since by throwing out a marginal unit of this commodity, the total profit decreases by a factor α. However, Propositions 1 and 2 still hold for games v_α in a neighborhood of v by continuity of the nucleolus and since as $\alpha \to 0$, $v_\alpha \to v$.

Appendix 1

Suppose that the players of P form a syndicate, while the players of Q stay unorganized. If we denote by P the syndicate, we have a $q+1$-player game with characteristic function[11]

$$v(S) = \begin{cases} \min(p, \lambda(s-1)) & \text{if } \; P \in S \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since the nucleolus is symmetrical, we need only to consider vectors in the set

$$X_{1,q} = \{x \in \mathbb{R}^{q+1} \mid x_P = \alpha,\ x_i = \beta \text{ for all } i \in Q,\ \alpha, \beta \geqslant 0,\ \alpha + \beta q = v(P \cup Q)\}.$$

[11] If not otherwise specified, capital letters refer to coalitions and small letters to the number of their elements.

Let $x = (\alpha, (\beta)^q)$ denote a typical element of $X_{1,q}$.

LEMMA 1. *Let* $S_0 \in \operatorname{argmax}_{S|P \in S} e(S, x)$ *and* $T_0 \in \operatorname{argmax}_{S|P \notin S} e(S, X)$ *and* $x \in X_{1,q}$. *The nucleolus is defined by the equality* $e(S_0, x) = e(T_0, x)$.

Proof. If

$$P \in S, \qquad e(S, x) = \min(p, \lambda(s-1)) - v(P \cup Q) + (q - s + 1)\,\beta. \quad (5)$$

and if

$$P \notin S, \qquad e(S, x) = -s\beta. \quad (6)$$

It is clear that $e(S, x)$ is increasing with β if $P \in S$ and is decreasing with β if $P \notin S$. Suppose that $e(S_0, x) > e(T_0, x)$. Define the vector $y = (\alpha', (\beta')^q)$ in $X_{1,q}$ such that $\beta' = \beta - \Delta$ with $\Delta = [e(S_0, x) - e(T_0, x)]/(q + t_0 - s_0 + 1)$. Then,

$$e(T_0, y) = e(S_0, y) = e(T_0, x) + t_0\Delta = e(S_0, x) - (q - s_0 + 1)\,\Delta.$$

So, $e(T_0, x) < e(T_0, y) = e(S_0, y) < e(S_0, x) = \max_S e(S, x)$, which proves that x cannot be the nucleolus. Q.E.D.

Now, consider two situations:

(i) $\lambda \leqslant p/q$. Then $p \geqslant \lambda(s-1)$ for all $s \in [1, q+1)$. Because $e(S, x) = 0$ for all $x \in X_{1,q}$ if $s = q + 1$, we can consider $s \leqslant q$. The excesses are

$$e(S, x) = \begin{cases} (q-s+1)(\beta - \lambda) & \text{if } P \in S, \\ -s\beta & \text{if } P \notin S. \end{cases}$$

Clearly, $e(S, x)$ is maximized at $s = q$ if $P \in S$ and at $s = 1$ if $P \notin S$. By Lemma 1, we know that the nucleolus is defined by $-\beta = \beta - \lambda$, i.e., $\beta = \lambda/2$, and $nu = (\lambda q/2, (\lambda/2)^q)$.

(ii) $\lambda \geqslant p/q$. Define the three sets,

$$E_1 = \{S \mid P \in S \text{ and } p \leqslant \lambda(s-1)\}$$

$$E_2 = \{S \mid P \in S \text{ and } p > \lambda(s-1)\}$$

$$E_3 = \{S \mid P \notin S\}.$$

By (5) and (6),

$$e(S, x) = \begin{cases} (q-s+1)\,\beta & \text{if } S \in E_1, \\ (s-1)\,\lambda - p + (q-s+1)\,\beta & \text{if } S \in E_2, \\ -s\beta & \text{if } S \in E_3. \end{cases}$$

Let $s \in [1, q]$. Then $e(S, x) \geqslant 0$ for all $S \in E_1$, the inequality being strict if $\beta \neq 0$. If $p \leqslant \lambda(q-1)$, $E_1 \neq \varnothing$, and because the nucleolus is an element of the core, we must have $\beta = 0$ and $nu = (p, (0)^q)$. Let $p > \lambda(q-1)$, i.e., $E_1 = \varnothing$. Then the excesses are maximized in E_2 for $s = q$ and in E_3 for $s = 1$. By Lemma 1, the nucleolus is defined by $-\beta = (q-1)\lambda - p + \beta$, and so, $\beta = (p - \lambda(q-1))/2$. Finally, $\alpha = p - q\beta$ implies that $\alpha = p(1 - q/2) + q(q-1)\lambda/2$.

In (i) and (ii), $\beta = \frac{1}{2}[v(P \cup Q) - v(P \cap Q) \setminus \{i\})]$, i.e., the syndicate always appropriates half of the marginal contributions of players of type Q. The same results are immediate for a syndicate of players of type Q.[12]

Appendix 2

We prove the Proposition 2 for the syndicate P (recall footnote 12). We will proceed in three steps, corresponding to the three parts of this proposition.

Step 1. Let $I \in \{P, Q\}$ and $\lambda \neq p/q$. I is advantageous iff I is strongly stable.

(Necessity) Suppose that $\lambda > p/q$, i.e., that P is not advantageous. Figures 1 and 2 imply that if $\lambda \in (p/q, p/(q-1))$, then $nu_i^v > nu_i^{v^{SP}}$ for all $i \in P$, where v denotes the market game in the competitive situation and v^{SP} the game when only the players of P form a syndicate. If $\lambda \geqslant p/(q-1)$, the players of type Q have an advantage to form a syndicate, and the syndicate P is not stable. In either case, because there exists a coalition structure for which a group of players gains by moving, the syndicate P is not strongly stable, which proves the necessity.

(Sufficiency) We suppose now that P is advantageous, i.e., that $\lambda < p/q$. We want to show that P is strongly stable. In order to do so, let us consider a structure $\pi = (P, \pi_Q)$, where P is the syndicate and π_Q is a given partition of Q. Let $T \subset P \cup Q$ and $\pi' = (\pi'_P, \pi'_Q)$ be any possible structure after the move of the players in T. We note that the change $\pi \to \pi'$ can be decomposed as $\pi \to \pi'' \to \pi'$ where $\pi'' = (P, \pi'_Q)$. I.e., only players in $T \cap Q$ move first, and then only players in $T \cap P$ move. Consider the change $\pi \to \pi''$, i.e., when only the players of $T \cap Q$ move. In order for π to be stable, no such move can be advantageous for the players in $T \cap Q$; in fact we would like that for any T, players in $T \cap Q$ be indifferent between π and

[12] Remark that $\min(p, \lambda(s-1)) = \lambda \min(p/\lambda, s-1)$. If $\lambda \lesseqgtr p/q$, $\mu = \lambda^{-1} \gtreqless q/p$, and, for the situation corresponding to the formation of the syndicate Q, the nucleolus is obtained by replacing p and q and q by p in the expression of the nucleolus in (i) and (ii) and multiplying the resulting expression by λ. For instance, if $\lambda \in [p/(q-1), \infty)$, $nu^{v^{SP}} = (p, (0)^q)$ and so $nu^{v^{SQ}} = ((0)^p, \lambda q)$ if $\lambda \in (0, (q-1)/p]$.

π''. Lemma 2 below establishes such a result: nu_P is constant for every partition π_Q and each player of Q obtains the same amount with the nucleolus irrespective of the syndicate to which he belongs and irrespective of the partition of Q. Lemma 3 establishes that by moving from π'' to π', all the players in $T \cap P$ strictly lose, i.e. that $nu_i^{v^{\pi''}} > nu_i^{v^{\pi'}}$, $i \in T \cap P$, and so that $nu_i^{v^{\pi'}} < nu_i^{v^{\pi}}$, $i \in T \cap P$, since Lemma 2 insures that the payoff of P in π is independent of π_Q. Consequently, the sufficiency is established. We now present the results.

LEMMA 2. *Let $\pi = (P, Q_1, ..., Q_m)$ be any partition of $P \cup Q$ with P as an element. Then, if $\lambda \leqslant p/q$,*

$$nu_P^{v^\pi} = \lambda q/2 \qquad \text{and} \qquad nu_{Q_i}^{v^\pi} = \lambda |Q_i|/2, \qquad i = 1, ..., m.$$

Proof. Consider a vector $x = (\alpha, \beta_1, ..., \beta_m) \in \mathbb{R}^{m+1}$ with $\alpha, \beta_i \geqslant 0$, $i = 1, ..., m$, and $\alpha + \sum_{i=1}^m \beta_i = q$; x is an imputation of the game (π, v^π). Let A^π be the set of permissible coalitions deduced from the partition π:

$$A^\pi = \{S \subset P \cup Q \mid (\text{there exist } i \in P \cup Q, j = 1, ..., m \text{ s.t. } i \in S \cap Q_j) \Rightarrow (k \in S \text{ for all } k \in Q_j)\}.$$

For a coalition $S \in A^\pi$, denote by I_S the subset of $\{1, ..., m\}$ such that $j \in I_S$ iff $Q_j \in S$. The excess of a coalition $S \in A^\pi$ with respect to x is then

$$e(S, x) = \begin{cases} \lambda \sum_{j \in I_S} |Q_j| - \alpha - \sum_{j \in I_S} \beta_j & \text{if } P \in S, \\ -\sum_{j \in I_S} \beta_j & \text{if } P \notin S. \end{cases} \tag{7}$$

Note that the core of the game v^π is nonempty since the vector $(0, \lambda|Q_1|, ..., \lambda|Q_m|)$ satisfies the core constraints. Once we notice that for $P \in S$, we can rewrite (7) as $e(S, x) = \sum_{j \notin I_S} (\beta_j - \lambda|Q_j|)$, it follows from the nonemptiness of the core that if $P \in S$, the maximum over A^B of the excesses is reached for the greatest value of $\sum_{j \in I_S} |Q_j|$. Rearrange the indexes in such a way that $|Q_1| \leqslant |Q_2| \leqslant \cdots \leqslant |Q_m|$, then the optimal value is $q - |Q_1|$ and the corresponding excess is

$$\begin{aligned} e(S, x) &= \lambda(q - |Q_1|) - \alpha - \beta_2 - \cdots - \beta_m \\ &= -\lambda|Q_1| + \beta_1. \end{aligned}$$

It is easily seen that the first m greatest excesses over A^π of coalitions with $P \in S$ are obtained for $\sum_{j \in I_S} |Q_j|$ equal to $q - |Q_1|$, $q - |Q_2|$, ..., $q - |Q_m|$ in this order.

From (7), it is also clear that the greatest value of the excesses when $P \notin S$ is reached for the coalition Q_{i^*} with the lowest payoff β_{i^*}. We show that this value must be β_1. Indeed, suppose $\beta_{i^*} < \beta_1$. Then by the same argument used for Lemma 1 we must have $-\beta_{i^*}=\beta_1-\lambda|Q_1|$ and so, $\beta_1+\beta_{i^*}=\lambda|Q_1|$. But if β_1 decreases to β'_1 and β_{i^*} increases to β'_{i^*} up to the point where $\beta'_1=\beta'_{i^*}$, both excesses $\beta'_1-\lambda|Q_1|$ and β'_{i^*} are inferior to $\beta_1-\lambda|Q_1|$ and $-\beta_{i^*}$ which proves that $\beta_1=\min_{i=1,\dots,m}\beta_i$ must be true for the nucleolus. Consequently, $\beta_1=\lambda|Q_1|/2$.

Because the second greatest excess when $P\in S$ is reached for $s-1=q-|Q_2|$ and when $P\notin S$ for $\beta_{i^*}=\min_{j\neq 1}\beta_j$, we find by the same argument as before that $\beta_2=\lambda|Q_2|/2$. Repeating this argument, $\beta_i=\lambda|Q_i|/2$ for all $i=1,\dots,m$. So,

$$\sum_{i=1}^{m} nu_{Q_i}^{v^\pi}=\lambda\sum_{i=1}^{m}|Q_i|/2=\lambda q/2 \qquad \text{and consequently} \qquad nu_P^{v^\pi}=\lambda q/2.$$

This result does not depend upon a particular choice of the partition π, moreover, if the syndicate shares equally its gains among its members, we have the result that each player in Q gets $\lambda/2$ in every syndicate and for every partition whenever P is a member of this partition. Q.E.D.

LEMMA 3. *Let $\pi=(P_1,\dots,P_r,Q_1,\dots,Q_m)$ be a given partition of $P\cup Q$ with $r\neq 1$. Then if $\lambda<p/q$, $nu_{P_i}^{v^\pi}<\lambda q|P_i|/(2p)$ for all $i=1,\dots,r$.*

Proof. Consider a vector $x=(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_m)\in\mathbb{R}^{r+m}$ such that $\alpha_i,\beta_j\geqslant 0$, $i=1,\dots,r$, $j=1,\dots,m$, and $\sum_{i=1}^r\alpha_i+\sum_{j=1}^m\beta_j=\lambda q$. Given the coalition structure π, the only permissible coalitions are those which can be written as a union of syndicates P_i and Q_j. I.e., if S is a permissible coalition, we must have two sets $I_S\subset\{1,\dots,r\}$, $J_S\subset\{1,\dots,m\}$ such that $S=\bigcup_{i\in I_S}P_i\cup\bigcup_{j\in I_S}Q_j$. Let A^π be the set of all permissible coalitions, and consider the following partition of A^π:[13]

$$A_+^\pi=\left\{S\in A^\pi \,\middle|\, \sum_{i\in I_S}|P_i|\geqslant\lambda\sum_{j\in J_S}|Q_j|\right\}$$
$$A_-^\pi=\left\{S\in A^\pi \,\middle|\, \sum_{i\in I_S}|P_i|<\lambda\sum_{j\in J_S}|Q_j|\right\}.$$

The excesses of the coalitions $S\in A_+^\pi$ with respect to the imputation x are

$$e(S,x)=\begin{cases}\lambda\sum_{j\in J_S}|Q_j|-\sum_{i\in I_S}\alpha_i-\sum_{j\in J_S}\beta_j & \text{if } S\in A_+^\pi,\\[1ex] \sum_{i\in I_S}|P_i|-\sum_{i\in I_S}\alpha_i-\sum_{j\in J_S}\beta_j & \text{if } S\in A_-^\pi.\end{cases} \tag{8}$$

[13] Note that for $\pi=(P,Q_1,\dots,Q_m)$, i.e., when $r=1$, $A_+^\pi=\{S\mid P\in S\}$ and $A_-^\pi=\{S\mid P\notin S\}$; indeed, $p\geqslant\lambda(s-1)=\lambda\sum_{j\in J_S}|Q_j|$ for all $s=1,\dots,q+1$.

Consider now the vector $y \in \mathbb{R}^{r+m}$ such that $\alpha_i = \lambda q |P_i|/(2p)$, $i = 1, ..., r$, and $\beta_j = \lambda |Q_j|/2$, $j = 1, ..., m$. By (8) we deduce that

$$e(S, y) = \begin{cases} \lambda\left(\sum_{j \in J_S} |Q_j| - q \sum_{i \in I_S} |P_i|/p\right)\Big/2 & \text{if} \quad S \in A^\pi_+, \\ \dfrac{2p - \lambda q}{2p} \sum_{i \in I_S} |P_i| - \lambda \sum_{j \in J_S} |Q_j|/2 & \text{if} \quad S \in A^\pi_-. \end{cases} \tag{9}$$

Consider the following imputation x:

$$\begin{aligned} \alpha'_i &= \alpha_i - \Delta |P_i|, & i &= 1, ..., r, \\ \beta'_j &= \beta_j + p\, \Delta |Q_j|/q, & j &= 1, ..., m, \end{aligned} \tag{10}$$

where $\Delta > 0$. The excesses with respect to x are

$$e(S, x) = e(S, y) + \Delta\left(\sum_{i \in I_S} |P_i| - p \sum_{j \in J_S} |Q_j|/q\right) \qquad \text{for all} \quad S \in A^\pi. \tag{11}$$

From (8), all the excesses with respect to the vector $(\underbrace{0, ..., 0}_{r}, \lambda|Q_1|, ..., \lambda|Q_m|)$ are nonpositive and so the core of the game v^π is nonempty.

Suppose now that for some coalition S in A^π_+ we have,

$$\sum_{j \in J_S} |Q_j| - q \sum_{j \in J_S} |P_i|/p \geqslant 0. \tag{12}$$

Then y does not belong to the core of (π, v^π) if this inequality is strict. Moreover, by (11), it is clear that both maximum excesses in A^π_+ and A^π_- decrease if $\Delta > 0$, so that nucleolus must be such that $nu_i^{v^\pi} < \lambda |P_i|/2$ for all $i = 1, ..., r$.

Suppose now that (12) is never true for $S \in A^\pi_+$. Then the following is true.

Claim. If (12) is not true, we have the equivalence $(S \in A^\pi_+) \Leftrightarrow (P \cup Q \backslash S \in A^\pi_-)$. Let $S \in A^\pi_-$. Then by definition,

$$\begin{aligned} \sum_{i \in I_S} |P_i| < \lambda \sum_{j \in J_S} |Q_j| \Leftrightarrow p - \sum_{i \in I_S} |P_i| &> p - \lambda \sum_{j \in J_S} |Q_j|, \\ \Rightarrow p - \sum_{i \in I_S} |P_i| &> \lambda\left(q - \sum_{j \in J_S} |Q_j|\right), \qquad \text{since} \quad p > \lambda q. \end{aligned}$$

I.e., $P \cup Q \backslash S \in A_+^\pi$. Now, if $S \in A_+^\pi$ and if (12) is not true,

$$q \sum_{i \in I_S} |P_i| > p \sum_{j \in J_S} |Q_j| \Leftrightarrow p - \sum_{i \in I_S} |P_i| < p - \sum_{j \in J_S} |Q_j| \; p/q$$
$$= \frac{p}{q}\left(q - \sum_{j \in J_S} |Q_j|\right),$$

which proves that $P \cup Q \backslash S \in A_-^\pi$ (otherwise (12) would be true for $P \cup Q \backslash S$). So, the claim is established.

If (12) does not hold, we may conclude that for all vectors given by (11), we have

$$\begin{aligned} e(S, x) &> e(S, y) \quad \text{if} \quad S \in A_+^\pi \\ e(S, x) &< e(S, y) \quad \text{if} \quad S \in A_-^\pi. \end{aligned} \tag{13}$$

If we show that $\max_{S \in A_+^\pi} e(S, y) < \max_{S \in A_-^\pi} e(S, y)$, Lemma 3 is proved since we have in this case by (13), $\max_{S \in A^\pi} e(S, x) < \max_{S \in A^\pi} e(S, y)$, which proves that the nucleolus must be a vector whose components are given by (11), i.e., with $nu_i^{v^\pi} < \lambda q |P_i|/(2p)$. From (9), the maximum of the excesses in A_+^π is reached for a coalition S such that $\sum_{I \in I_S} |P_i|$ is minimum and $\sum_{j \in J_S} |Q_j|$ is maximum. Similarly, the maximum of the excesses in A_-^π is reached for a coalition S such that $\sum_{i \in I_S} |P_i|$ is maximum and $\sum_{j \in J_S} |Q_j|$ is minimum. The claim establishes that these two coalitions are complementary, i.e., that S^* maximizes $e(S, x)$ in A_-^π when $P \cup Q \backslash S^*$ maximizes $e(S, x)$ in A_+^π. So, if S^* is this coalition, we have

$$e(S^*, y) = \max_{S \in A_-^\pi} e(S, y) = \frac{2p - \lambda q}{2p} \sum_{i \in I_{S^*}} |P_i| - \frac{\lambda}{2} \sum_{j \in J_{S^*}} |Q_j|$$

and

$$e(P \cup Q \backslash S^*, y) = \max_{S \in A_+^\pi} e(S, y) = \frac{\lambda}{2}\left[q - \sum_{j \in J_{S^*}} |Q_j| - \frac{q}{p}\left(p - \sum_{i \in I_{S^*}} |P_i|\right)\right]$$
$$= \frac{\lambda q}{2p} \sum_{i \in I_{S^*}} |P_i| - \frac{\lambda}{2} \sum_{j \in J_{S^*}} |Q_j|.$$

Since $\lambda < p/q \Leftrightarrow \lambda q/2p < (2p - \lambda q)/2p$, we have $\max_{S \in A_-^\pi} e(S, y) > \max_{S \in A_+^\pi} e(S, y)$ which proves Lemma 3. Q.E.D.

Step 2. By Lemma 3, every syndicate $T \subset P$, $T \neq P$ cannot be strongly stable. Indeed for any partition $\pi = (\pi_P, \pi_Q)$ with $T \in \pi_P$, it is true that $nu_i^{v^\pi} < nu_i^{v^{\pi'}}$ where $i \in P \backslash T$ and $\pi' = (P, \pi_Q)$. Consequently (2) is established.

Step 3. When $\lambda = p/q$, (9) implies that the vector y defined by $\alpha_i = |P_i|/2$ and $\beta_j = p|Q_j|/2q$, $i = 1,..., r$, and $j = 1,..., m$ is such that for $S \in A_+^\pi$, $e(S, y) = e(P \cup Q \backslash S, y)$. So for every vector $x \neq y$, we must have $\max_{S \in A^\pi} e(S, x) > \max_{S \in A^\pi} e(S, y)$, and y is the nucleolus of the game v^π. Because this is true for every partition π, we conclude that every syndicate is strongly stable when $\lambda = p/q$. This concludes the proof of Proposition 2.

Q.E.D.

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