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# Monotone Matching in Perfect and Imperfect Worlds

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We study frictionless matching in large economies with and without market imperfections, providing sufficient conditions for monotone matching that are weaker than those previously known. Necessary conditions, which depend on a key analytical object we call the *surplus function*, are also offered. Changes in the surplus yield valuable information about the comparative statics of matching patterns across environments. We apply our framework to some examples adapted from the literature, accounting for and extending several comparative-static and welfare results. We also explore the dependence of the matching pattern on the type distribution.

## 1. INTRODUCTION

Ever since Roy (1950) and Tinbergen (1951) used them to study the distribution of earnings, frictionless matching models have been adapted to a wide range of problems.<sup>1</sup> Early applications tended to consider environments with perfect markets: the only departure from standard Arrow–Debreu assumptions was the presence of an indivisibility in agents' characteristics that make a matching problem relevant. But as many recent applications—including, for example, international trade, local public finance, or organizational design (Legros and Newman (1996), Casella and Rauch (1997), Farrell and Scotchmer (1998), Sadoulet (1998))—indicate, matching models are also natural and appropriate vehicles for studying environments with market imperfections.

Among the most important and robust insights of the early literature was a fundamental monotonicity result. When agents' characteristics are complementary—their joint output is a supermodular function of the characteristics—there is positive assortative matching (PAM): *regardless of the distribution of types*, more able individuals are assigned to more productive tasks or to more able individuals. Monotone matching patterns of this kind are compelling for their computational expedience as well as their empirical appeal, and in the minds of most economists, the connection among efficiency, PAM and complementarities is probably *the* main idea in the matching literature.

Yet it is less than clear how far this connection carries over to the more general environments that have attracted recent attention. Some investigations of credit market imperfections suggest that complementarities in the production technology alone need not entail PAM (Legros and Newman (1996), Sadoulet (1998)). Similarly, restrictions on the way output is shared that arise

1. Sattinger (1993) provides a fine survey of the classic references as well as some more recent literature.

for technological or incentive reasons may drastically alter the pattern of matching and its efficiency (Becker (1973), Farrell and Scotchmer (1998)). Even without imperfections, there are natural examples of technologies that certainly seem to exhibit complementarity and yet fail to satisfy supermodularity (Kremer and Maskin, 1996).

Moreover, the presence of market imperfections leads to the possibility that matches may not be efficient, at least in the sense of maximizing social surplus. But without analytical tools for computing equilibrium matching patterns, it is difficult to evaluate the efficiency of equilibrium matches and assess the role of policy designed to affect matching outcomes.

Our purpose here and in our companion paper (Legros and Newman, 2002a) is to provide such tools by developing sufficient and necessary conditions for monotone matching in some of these more general environments. These conditions, which like the classical supermodularity condition, apply independently of the distribution of characteristics, facilitate computation of equilibria (including those in classical environments) and help with an assessment of the impact of imperfections on matching. The present paper is devoted to the analysis of the transferable utility case, which, as we shall see, includes a number of imperfect-market examples. For an analysis of the non-transferable case see Legros and Newman (2002a).

Two principal themes emerge from the analysis. First, several monotone matching patterns that have appeared in the literature and that are incompatible with supermodularity of the joint output function can be understood as consequences of other, weaker, monotone difference conditions. Second, a great deal about the nature of the equilibrium match can be gleaned simply by studying an object we call the *surplus*: defined as the difference between the total payoff to a heterogeneous partnership and the average of the partners' *segregation payoffs* (the equilibrium payoffs to individuals in one-type economies), it measures the potential gains to a heterogeneous match.

We use our apparatus to analyse several extended examples adapted from the recent literature. These are introduced in the next section, where we point out where the existing apparatus falls short. As we present our results in Sections 4.1–5, we return to the relevant examples to illustrate their use.

## 2. EXAMPLES AND ISSUES

The version of the matching model we shall consider is one in which agents of varying abilities (or other real-valued characteristics such as wealth, skill or productivity) form two-person partnerships. When one agent of type  $a$  matches with an agent of type  $b$ , they produce a positive “output”  $h(a, b)$  which can be divided arbitrarily between them. The function  $h$  is usually assumed to be symmetric: ( $h(a, b) = h(b, a)$ ) and increasing in both arguments.

The famous result, proved for completeness in Proposition 3 below, is that if  $h$  has a non-negative cross partial derivative everywhere (more generally, if it is supermodular<sup>2</sup>), then in competitive equilibria (or core allocations, our focus here), there will be a particularly strong form of assortative matching which we call *segregation*: everyone matches with someone identical to himself (the model in Kremer (1993) has this property). If instead  $h$  is submodular, there is *negative assortative matching* (NAM): the higher is the ability of one partner, the lower is the ability of the other; see Proposition 7.

Segregation and NAM are examples of matching patterns that are *monotone*: in both cases,  $a$ 's partner's type is monotonic in  $a$ .

2. A real-valued function  $h$  on  $D \subset \mathbb{R}^2$  is *supermodular* if for any  $x, y \in D$ ,  $h(x \wedge y) + h(x \vee y) \geq h(x) + h(y)$ , where we denote by  $x \wedge y$  the component-wise minimum of  $x$  and  $y$ :  $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ ; similarly,  $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\})$ . *Submodularity* holds if  $h$  satisfies  $h(x \wedge y) + h(x \vee y) \leq h(x) + h(y)$  on  $D$ .

**Limitations of the supermodularity condition.** As useful as these results are, they have serious limitations. Supermodularity is only sufficient for segregation: there are economies without supermodular production functions that are nevertheless always segregated in equilibrium. The gap between sufficiency and necessity is easily bridged if one considers the *surplus* rather than the joint payoff, and we will provide a distribution-free condition for segregation (Proposition 4) that is necessary as well as sufficient.

By the same token, other forms of monotone matching have appeared in the literature, and they are necessarily inconsistent with super- or submodularity. But as we shall see, environments without supermodularity need not be any less amenable to systematic analysis.

**Post-match task assignment.** For instance, suppose, as in Kremer and Maskin (1996), that the partners might be assigned to either task after the match occurs, but that the two tasks do not contribute symmetrically to the joint output. We could represent this situation by supposing that the production function is of the form  $\max\{a^\theta b^{1-\theta}, b^\theta a^{1-\theta}\}$ , where  $\theta \in (\frac{1}{2}, 1)$ . The ability of the partner assigned to the more “important” task is raised to the  $\theta$  power; it is always optimal to assign the more able partner to this task. This production function is not supermodular.<sup>3</sup> Existing results on matching are therefore silent on this simple extension of the basic model. We shall provide a new sufficient condition for PAM (“weak increasing differences” or WID; see Proposition 5) which subsumes this model, and we go on to analyse properties of the model in Section 4.1.1.

**Imperfect credit markets.** These present a different sort of problem. They may introduce non-transferabilities, but this is not fundamental. Rather, they tend to undermine, even overwhelm, the complementarity of the production technology in ways that make computation of the matching pattern a challenge.

Modify the standard production model by supposing that a fixed amount  $k > 0$  of capital with unit cost one is required for production to take place. Once this is invested, gross output depends on the ability of the firm’s members and is equal to  $ab$ , where all abilities exceed  $\sqrt{k}$ . All individuals have zero wealth, and therefore every partnership must access a capital market in order to finance their firm.

This market is imperfect, however: the output of a firm must exceed  $\phi k$ ,  $\phi \geq 1$ , in order for financing to be possible.<sup>4</sup> The joint output for a pair  $(a, b)$  can then be written as

$$h(a, b; \phi) = \begin{cases} ab - k, & \text{if } ab \geq \phi k \\ 0, & \text{if } ab < \phi k. \end{cases}$$

A perfect capital market corresponds to  $\phi = 1$ . In this case, the economy will be segregated by ability, since  $h$  is symmetric and supermodular. This outcome is independent of the initial distribution of types.

3. To see this, note that supermodularity of  $h(\cdot, \cdot)$  implies that for points of the form  $(a, b)$  and  $(b, a)$ ,  $a > b$ , we have  $h(a, a) + h(b, b) \geq h(a, b) + h(b, a)$ , or in this case that  $a + b \geq 2a^\theta b^{1-\theta}$ . Letting  $b = \lambda a$  for  $0 < \lambda < 1$ , this is equivalent to the requirement that  $1 + \lambda \geq 2\lambda^{1-\theta}$ , which obviously fails for  $\lambda$  near 1. Thus, this production function fails the supermodularity test near the diagonal. (Since the cross partial derivative is positive wherever it exists,  $h$  is clearly not submodular, either.)

4. This admittedly stark version of a capital market imperfection can be derived by supposing that the partners in the firm, upon having to repay, may renege on their debt and escape with probability  $\pi$  a punishment which brings their income to zero. Lenders will make loans of size  $k$  only to those firms whose output  $h$  will exceed  $k/(1 - \pi)$ , since only for these firms is repaying, which yields a payoff of  $h - k$ , more attractive than renegeing, which yields  $\pi h$ . Thus,  $\phi \equiv \frac{1}{1-\pi} = 1$  corresponds to  $\pi = 0$ ; with larger values of  $\pi$  escape becomes more likely, until with  $\pi = 1$ , the market shuts down altogether ( $\phi = \infty$ ).

As  $\phi$  increases, the market becomes less efficient, excluding more and more types from producing positive output on their own. It is straightforward to verify that in these cases, the payoff function  $h$  is not supermodular in types and also violates the weaker WID condition. Yet the model remains one of transferable utility, since whatever output a coalition does produce can be shared arbitrarily between the partners. The question then arises, does monotone matching result regardless of the distribution of types? It turns out that this model violates our conditions and therefore we can find distributions for which the match is sometimes monotone, and sometimes non-monotone. The example illustrates how the comparative statics of the surplus across environments, and particularly how they vary across types, can reveal a lot about changes in matching patterns.

**The dependence of the match on distribution.** The dependence of the pattern of matching on the distribution of types has attracted attention recently (Kremer and Maskin, 1996). In fact, it is clear that unless the economy is always segregated, the match typically must depend on the distribution, in the sense that the correspondence  $\mathcal{M}(a)$  which sends a type  $a$  into the type(s) with which it matches will not be invariant to the distribution. Of course, requiring that  $\mathcal{M}(a)$  be invariant is very demanding.

Instead one might only require that monotonicity of the match be preserved. Indeed, what is most remarkable about the classical supermodularity result is precisely that it is independent of the distribution. One need not compute the equilibrium to obtain the prediction, and indeed, knowing that the outcome is monotonic often helps in computing the equilibrium! Thus, the kind of sufficiency conditions one might seek first for broader classes of models are those that are distribution-free. In order to assess how far these conditions can be pushed, though, one wants necessary conditions.

As well as yielding insights into models in which they apply, these conditions can be useful for understanding models in which they don't. For instance, in Kremer and Maskin (1996), the main result concerning the effect of distribution on the degree of segregation is easily understood as a consequence of the shape of the surplus functions. That model always has monotone matching, however, and the dependence of matching on distribution is reflected by changes in cardinal measures of the matching correspondence. The more striking dependence of matching on distribution that occurs in our imperfect financial market example is a consequence of the violations of the necessary and sufficient conditions for monotonic matching that we develop below.

### 3. THEORY

#### 3.1. Notation

The economies we study have a continuum of agents and a “type space”  $A$  that is a compact subset of the real line. There is a distribution  $T$  of types, which may be continuous or discrete. Either way, we think of there being a continuum of agents of every type.

The object of analytical interest to us is the utility possibility set for each possible coalition, *i.e.* the characteristic function of a cooperative game representing the matching problem. We follow much of the literature and restrict our attention to matches of size two (some of our results generalize straightforwardly to multiperson matches, as we will indicate). Since we are considering transferable utility, the set of feasible payoffs of a pair consisting of a type  $a$  and type  $b$  can be written

$$V(a, b) = \{(u_1, u_2) \in \mathbb{R}_+^2 | u_1 + u_2 \leq h(a, b)\}.$$

(The restriction to matches of size two can be formalized by supposing that every unmatched individual gets a payoff less than or equal to zero, and that for any larger coalition of individuals,

the total payoff is less than or equal to  $\int_j h(a_j, b_j)$ , where  $j$  indexes the elements of a partition of the coalition—possibly less one agent—into sets of size two with types  $(a_j, b_j)$ .)

The notation reflects two assumptions, which we maintain throughout: (1) the payoff possibilities depend only on the types of the agents and not on their individual identities; and (2) the utility possibilities of the pair of agents do not depend on what other agents in the economy are doing, *i.e.* there are no externalities across coalitions.<sup>5</sup>

In a typical application, all agents are expected income maximizers who can feasibly share the output of their joint production in any way. The level of output they can generate depends on their type according to a (possibly stochastic) “production function”  $h(a, b)$ . To help ensure existence of equilibrium, this function is assumed to be upper semicontinuous in the types. Note that  $h$  may be generated in part by choices made by the partners.

Define the *segregation payoff* of type  $a$  as  $u(a) = \frac{h(a,a)}{2}$ . The segregation payoff has the interpretation of the minimum utility that an agent can expect to get: if two agents of a particular type get less than this, they can always match together and share the output equally. It will often be convenient to analyse economies using a modified characteristic function that captures the notion of the potential gains from heterogeneous matching. Let

$$\sigma(a, b) \equiv \max\{0, h(a, b) - \frac{1}{2}[h(a, a) + h(b, b)]\},$$

and call it the *surplus function*. Observe that  $\sigma(a, a) = 0$  for all  $a$ .

### 3.2. Equilibrium

We use the core as our equilibrium concept.<sup>6</sup> The equilibrium consists of a matching correspondence  $\mathfrak{M}^* : A \rightrightarrows A$  that specifies the way types are matched to each other and a payoff allocation  $u^* : A \rightarrow \mathbb{R}$  denoting the payoff to each type;<sup>7</sup> it will satisfy two key properties. First is a “measure consistency” condition, which says that the mass of “first partners” equals that of the “second partners” (without this, it would be possible to match, say, a one-half measure of men one-to-one to a unit measure women). Second, a no-blocking condition is satisfied, *i.e.* for all  $a$  and  $b$ , there does not exist a payoff vector  $(u(a), u(b))$  with  $u(a) + u(b) \leq h(a, b)$  such that  $u(a) > u^*(a)$  and  $u(b) > u^*(b)$ . Since equilibrium payoffs must be feasible, this implies  $u^*(a) + u^*(b) = h(a, b)$  for every matched pair. See the Appendix for details.

We first note that *all equilibria are constrained Pareto efficient*. If there were a Pareto improvement, then the grand coalition could block the equilibrium payoff; but since the grand coalition cannot achieve anything more than what two-person coalitions can achieve, a two-person coalition could also block, and this would violate the definition of an equilibrium.

Since there is transferable utility, something much stronger can be asserted, namely that the equilibrium match will maximize the aggregate net output (this includes in particular our imperfect financial market example). In this case, if there are any choice variables, any pair of matched agents will choose them so as to maximize their joint output. Call this maximized value  $h(a, b)$  when an  $a$  matches with a  $b$ . Observe that if  $a$  and  $b$  are two types that are not matched to each other in equilibrium, then  $u^*(a) + u^*(b) \geq h(a, b)$ , else the pair  $(a, b)$  would block. Now, if the equilibrium matching pattern fails to maximize aggregate net output, there is another measure consistent match that generates a higher aggregate; in this alternate match, there is at

5. Of course the equilibrium payoffs in one coalition *will* generally depend on the other coalitions.

6. The facts that there is a continuum of agents and that the only coalitions that matter are of size two at most technically make the core here a special case of the  $f$ -core. See Kaneko and Wooders (1986) for definitions. Their paper and related results in Gretsky, Ostroy and Zame (1992) guarantee existence of equilibrium for all cases we consider.

7. The notation reflects the fact that in these environments equilibrium has the equal treatment property: all agents of a given type will get the same utility.

least one type  $a$  matched to a  $b'$  such that  $u^*(a) + u^*(b') < h(a, b')$ , or the aggregate could not be higher. But then the pair  $(a, b')$  would have blocked the original equilibrium. A similar argument can be made for the aggregate surplus and we have the following proposition.

**Proposition 1.** *In equilibrium (i) the match is efficient in the sense that given the type distribution, it maximizes aggregate net output; and (ii) aggregate surplus is also maximized.*

The optimality of equilibrium (i) under transferable utility is, of course, well known; what we want to emphasize here is that certain market imperfections can still be treated under the rubric of transferable utility and therefore lead to efficient outcomes. We will return to this point in Section 5. As we will also show there, result (ii) can be useful in computations.

### 3.3. Descriptions of equilibrium matching patterns

A match or assignment is a measurable correspondence

$$\mathfrak{M}^* : A \rightrightarrows A.$$

$\mathfrak{M}^*$  is symmetric:  $a \in \mathfrak{M}^*(b)$  implies  $b \in \mathfrak{M}^*(a)$ . Let

$$\bar{A} = \{a \in A : \exists b \in \mathfrak{M}^*(a) : b \leq a\}$$

be the set of larger partners (obviously,  $\bar{A}$  depends on  $\mathfrak{M}^*$ , but we suppress this dependence in the notation). Symmetry of  $\mathfrak{M}^*$  implies that the correspondence  $\mathfrak{M}$

$$\mathfrak{M} : \bar{A} \rightrightarrows A, \quad \text{where } b \in \mathfrak{M}(a) \iff b \in \mathfrak{M}^*(a) \& b \leq a,$$

completely characterizes the assignment. Note that the graph of  $\mathfrak{M}$  is the portion of the graph of  $\mathfrak{M}^*$  that is on or below the  $45^\circ$  line. The coalitions generated by  $\mathfrak{M}^*$  can then be written as ordered pairs  $(a, b) \in \bar{A} \times \mathfrak{M}(\bar{A})$ . Our descriptions of matching patterns will be in terms of the properties of the graph of  $\mathfrak{M}$ .

We will be focused on patterns in which  $\mathfrak{M}$  is a monotone correspondence. For two sets  $M$  and  $M'$ , we write  $M \geq M'$  whenever  $[a \in M \& b \in M'] \Rightarrow [a \geq b]$ .

**Definition 1.** Matching is monotone if

(i) for all  $a, b \in \bar{A}$ ,  $[a > b \Rightarrow \mathfrak{M}(a) \geq \mathfrak{M}(b)]$

or

(ii) for all  $a, b \in \bar{A}$ ,  $[a > b \Rightarrow \mathfrak{M}(a) \leq \mathfrak{M}(b)]$ .

In case (i), the graph of  $\mathfrak{M}$  must be non-decreasing while in case (ii) the graph of  $\mathfrak{M}$  must be non-increasing. For this reason, we refer to case (i) as PAM and to case (ii) as NAM.<sup>8</sup>

8. The terminology PAM and NAM appears to have originated in “two-sided” matching models (*e.g.* the marriage market model of Becker (1973)), in which agent types include a gender as well as a real-valued attribute. By ordering the types in a two-sided model lexicographically, first by gender and then by ability, our definitions of PAM and NAM include the two-sided notions as special cases, with  $\bar{A}$  corresponding to one of the sides.

We are aware of only one other attempt to give a formal definition of PAM in one-sided models, namely that by Shimer and Smith (2000), who also proceed through descriptions of the correspondence  $\mathfrak{M}$  (equivalently  $\mathfrak{M}^*$ ). For PAM they require that the graph of  $\mathfrak{M}^*$  form a lattice. This is a useful definition for the problem they are studying, namely matching under search frictions, but it is too weak for the frictionless case. The class of models they consider leads to segregation in the absence of search frictions.

Note that PAM and NAM are not “opposites”: there are many matching patterns that are neither PAM nor NAM. Indeed, NAM is really more stringent than PAM: it rules out two sorts of matches among the types  $a > b \geq c > d$ , namely  $\{a, c\}, \{b, d\}$  and  $\{a, b\}, \{c, d\}$  while PAM rules out only matches  $\{a, d\}, \{b, c\}$ . The stringency of NAM follows from a kind of boundary condition that is entailed in its definition: if  $\bar{a}$  is the highest type and  $\underline{a}$  the lowest, then under NAM we necessarily have  $\bar{a}$  matched with  $\underline{a}$ , which does much to restrict what the rest of the matches look like.

A simple and strong form of PAM occurs when each agent matches only with someone like himself, a condition we refer to as *segregation*.

**Definition 2.** An equilibrium satisfies *segregation* if  $\mathfrak{M}(a) = \{a\}$  for all  $a$ .

That is, segregation occurs when the graph of  $\mathfrak{M}$  is a subset of the  $45^\circ$  line. It is an extreme kind of equilibrium outcome, since it precludes any matches in which partners are different from each other. If the match is not segregated, we shall say that it is *heterogeneous*. If almost every matched pair contains partners who are not identical, we shall say that it is *perfectly heterogeneous*.

In case the distribution of types  $T(\cdot)$  is continuous, we can sharpen the characterization of the matching correspondence. First of all, we note that if the matching is monotone,  $\mathfrak{M}$  is almost everywhere single-valued. A proof is given in the Appendix.

**Proposition 2.** Suppose that  $T$  is continuous. If the equilibrium match is monotone then for almost every  $a \in T$ ,  $\mathfrak{M}(a)$  is a singleton.

When  $T(\cdot)$  is continuous with median  $a_m$ , NAM implies that partners match “across the median”, i.e. that  $a > a_m \Leftrightarrow a_m > \mathfrak{M}(a)$ . Moreover, it is straightforward to show that by measure consistency  $T(a) + T(\mathfrak{M}(a)) = 1$ , where the singleton property ensures this expression is well defined. Such a characterization in terms of  $T$  is not possible for PAM. Nevertheless, there is a form of PAM that is analogous to NAM in the sense that partners also match across the median and can be characterized in terms of the distribution  $T$ ; we call it median matching.

**Definition 3.** An equilibrium displays *median matching* when it satisfies PAM and for all  $a \in [a_m, \bar{a}]$ ,  $T(a) - T(\mathfrak{M}(a)) = \frac{1}{2}$ .

Observe that median matching precludes, for  $a > b \geq c > d$ , pairings of the form  $\{a, b\}, \{c, d\}$  (except in the case  $b = c = a_m$ ), as well as  $\{a, d\}, \{b, c\}$ .

For four types  $a > b > c > d$ , the types of matching consistent with the different definitions are represented in Figure 1.

For brevity, we will say that an economy is *segregated (positively, negatively matched)*, if all equilibria are payoff equivalent to one with segregation (positive, negative matching).

#### 4. SUFFICIENT CONDITIONS

##### 4.1. Positive assortative matching

For completeness, we first state and prove the well-known result leading to segregation, which as we have pointed out is a strong form of PAM.

**Proposition 3.** If the symmetric production function  $h(a, b)$  is supermodular, the economy is segregated.

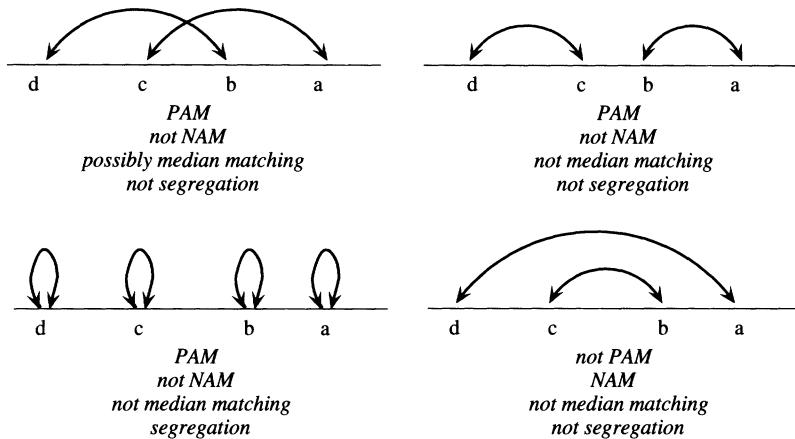


FIGURE 1

*Proof.* If  $h$  is supermodular, it satisfies the inequality  $h(x \vee y) + h(x \wedge y) \geq h(x) + h(y)$ ; putting  $x = (a, b)$  and  $y = (b, a)$  and using symmetry then implies that  $h(a, b) - \frac{1}{2}[h(a, a) + h(b, b)] \leq 0$ . An allocation that involves a heterogeneous match that is not payoff equivalent to segregation therefore involves at least one type receiving less than its segregation payoff, and the allocation would then be blocked. ||

Note that under these conditions, the surplus function  $\sigma(\cdot, \cdot)$  is identically zero. In fact, this is a much weaker sufficient condition for segregation:

**Proposition 4.** *If  $\sigma(a, b) = 0$  for all  $a, b \in A$ , the economy is segregated.*

To see this, note that any heterogeneous outcome that is not payoff equivalent to segregation would be blocked by at least one type which is receiving less than its segregation payoff. Note that  $\sigma(a, b) = 0$  is equivalent to having vector of segregation payoffs  $(\underline{u}(a), \underline{u}(b))$  being “outside” the interior of  $V(a, b)$ . Not only is this weaker than imposing supermodularity on the production function, but it can be sometimes easily verified in cases where the production function does not satisfy standard properties.

*Example.* Let  $A = [1, 2]$  and  $h(a, b) = \sqrt{a} + \sqrt{b} - \frac{1}{3} \max[a^{2/3}b^{1/3}, b^{2/3}a^{1/3}]$ . It is straightforward to verify that  $h_a$  and  $h_b$  are positive wherever they exist (which is everywhere except on the diagonal). As we saw in the post-match task assignment example above,  $\{a^{2/3}b^{1/3}, b^{2/3}a^{1/3}\}$  is neither super- nor submodular, so neither is  $h$ . Yet  $h(a, b) - \frac{1}{2}[h(a, a) + h(b, b)] \propto a + b - 2 \max[a^{2/3}b^{1/3}, b^{2/3}a^{1/3}] \leq 0$  on  $A^2$ . Thus,  $\sigma(a, b) = \max\{0, h(a, b) - \frac{1}{2}[h(a, a) + h(b, b)]\} = 0$  there, and the economy is segregated.

It is well known that in the present context, supermodularity is equivalent to *increasing differences* (ID): whenever  $a > b$  and  $c > d$ , we have  $h(a, c) - h(a, d) \geq h(b, c) - h(b, d)$ . This is actually a fairly strong condition to impose on the production function. As we pointed out before, it rules out some fairly natural cases, and in particular does not allow for any sort of heterogeneous matching.

Notice that to satisfy ID,  $h(\cdot, \cdot)$  must be checked against six possible permutations of  $a, b, c$  and  $d$ . A weaker requirement for  $h(\cdot, \cdot)$  only involves looking at quadruples of types arranged in the order  $a > b \geq c > d$ .

*Definition 4.* The function  $h : A^2 \rightarrow \mathbb{R}$  satisfies WID on  $A$  if for any four elements  $a, b, c, d$  of  $A$ , where  $a > b \geq c > d$ ,

$$h(b, c) - h(b, d) \leq h(a, c) - h(a, d)$$

or

$$h(b, c) - h(c, d) \leq h(a, b) - h(a, d).$$

Another way to see that WID is a weakening of ID (and supermodularity) is to note that the latter implies that

$$h(a, c) - h(a, d) \geq h(b, c) - h(b, d) \quad \text{and} \quad h(a, b) - h(a, d) \geq h(c, b) - h(c, d)$$

whenever  $a > b \geq c > d$ . (Coupled with the symmetry assumption, these two inequalities plus  $h(a, b) - h(a, c) \geq h(b, d) - h(c, d)$  are equivalent to ID.) When  $h(\cdot, \cdot)$  is differentiable, it can be shown that WID and ID are equivalent to each other, and it is well known that this in turn requires that the cross partial derivative of  $h$  is non-negative.

Our first result follows directly from the observation that for PAM, it is really only the order  $a > b \geq c > d$  that matters: we need simply to ensure that the negatively matched arrangement  $\{a, d\}$  and  $\{b, c\}$  cannot happen in equilibrium (unless it is payoff equivalent to a positive match). We therefore have the following proposition.

**Proposition 5.** *If  $h(\cdot, \cdot)$  is symmetric and satisfies WID on  $A$ , the economy is positively matched for all distributions of types with support in  $T$ .*

*Proof.* Let  $a > b \geq c > d$ . If, contrary to the conclusion,  $\{a, d\}$  and  $\{b, c\}$  are matched, we must have  $h(a, d) + h(b, c) \geq h(a, c) + h(b, d)$  and  $h(a, d) + h(b, c) \geq h(a, b) + h(c, d)$ , else the negative match could be blocked. If the match is not payoff equivalent to PAM, then one or both of these inequalities must be strict, contradicting WID.  $\parallel$

*Example.* The production function in the post-match task assignment model,  $h(a, b) = \max(a^\theta b^{1-\theta}, b^\theta a^{1-\theta})$  for any  $\theta \in [\frac{1}{2}, 1]$ , satisfies ID if and only if  $\theta = \frac{1}{2}$ , in which case the economy is segregated. Otherwise, the function is neither differentiable (the limits of  $\frac{1}{\epsilon}[h(a + \epsilon, a) - h(a, a)]$  as  $\epsilon \uparrow 0$  and  $\epsilon \downarrow 0$  are not equal) nor supermodular (as we pointed out earlier). However, it does satisfy WID: for positive  $a > b \geq c > d$ , we have  $a^\theta(c^{1-\theta} - d^{1-\theta}) > b^\theta(c^{1-\theta} - d^{1-\theta})$ . Thus the post-match task assignment economy always has PAM.

Because segregation is a form of PAM, we did not have to rule it out in order for Proposition 5 to be true. But for other patterns of matching with which segregation is inconsistent, we shall require an auxiliary condition:

*Definition 5.* Condition H is satisfied if  $h(a, b) > \frac{1}{2}(h(a, a) + h(b, b))$  whenever  $a \neq b$ .

If H is satisfied, there will always be perfectly heterogeneous matching in equilibrium, provided there is more than one type in the economy. More succinctly, perhaps, we get perfectly heterogeneous matching if the surplus is strictly positive off the diagonal.

**4.1.1. Median matching in one-sided models.** We turn next to median matching. The logic is similar: one needs a condition which ensures that when types  $a > b \geq c > d$  are arranged in a pattern consistent with median matching, the total payoffs are higher than they are under any other arrangement. The key condition is a strengthening of WID:

*Definition 6.* The function  $h : A^2 \rightarrow \mathbb{R}$  satisfies Condition M if for  $a > b \geq c > d$ , we have

$$h(b, c) - h(b, d) \leq h(a, c) - h(a, d) \quad \text{and} \quad (1)$$

$$h(b, d) - h(c, d) \geq h(a, b) - h(a, c), \quad (2)$$

with strict inequality whenever  $b > c$ .

That this is indeed a strengthening of WID should be clear.<sup>9</sup> But unlike WID, it is not a weakening of supermodularity, since the latter requires  $h(b, d) - h(c, d) \leq h(a, b) - h(a, c)$  instead of (2).

Condition M, together with the heterogeneity condition (Condition H) are sufficient for median matching.

**Proposition 6.** *Suppose that Conditions M and H hold, that  $h$  is symmetric and that the type distribution is continuous. Then the unique equilibrium outcome is median matching.*

*Proof.* For  $a \in [\underline{a}, \bar{a}]$ , let  $m(a)$  denote an element of  $\mathcal{M}^*(a)$  (by Proposition 2,  $m(a) = \mathcal{M}^*(a)$  for almost all  $a$ ). Now suppose that  $m(\bar{a}) < a_m$ . Since  $h$  satisfies WID, we have PAM, which implies that  $m(a) \leq m(\bar{a})$  for  $m(\bar{a}) < a < \bar{a}$ . But this violates measure consistency, since more than one-half the population is matching with less than one-half the population. Thus  $m(\bar{a}) \geq a_m$ , and since Condition H holds,  $\bar{a} > m(\bar{a})$ . A similar argument establishes that  $\underline{a} < m(a) \leq a_m$ . Suppose that  $m(\bar{a}) > a_m$  or  $m(\underline{a}) < a_m$ . Then (2) implies directly  $h(\bar{a}, m(\bar{a})) - h(\bar{a}, m(\underline{a})) < h(\underline{a}, m(\bar{a})) - h(\underline{a}, m(\underline{a}))$ ; thus either  $\{\bar{a}, m(\underline{a})\}$  or  $\{\underline{a}, m(\bar{a})\}$  could improve upon the equilibrium, a contradiction. Therefore  $m(\underline{a}) = m(\bar{a}) = a_m$ . To complete the argument for the remaining types, note that if  $T(a) - T(m(a)) > \frac{1}{2}$  for  $a > a_m$ , then the measure of agents between  $a_m$  and  $a$ , who by PAM are matching with agents between  $\underline{a}$  and  $m(a)$ , exceeds that of the latter set, which violates measure consistency. A similar violation of measure consistency occurs if  $T(a) - T(m(a)) < \frac{1}{2}$ . ||

*Remark 1.* A somewhat weaker result can be established if one dispenses with Condition H and instead imposes continuity on the production function. Then under the remaining hypotheses of Proposition 6, median matching is *an* equilibrium outcome, albeit not necessarily the only one. The proof depends on noting that continuity and (2) imply that segregation generates no higher output than heterogeneous matching (take limits as  $b \rightarrow a$  and  $c \rightarrow d$ ); from there the argument is similar to that of the proposition.

Several observations are in order. First, the proof of this result makes heavy use of the fact that the matching pattern is monotone, illustrating that in matching models, as in many other contexts, monotonicity greatly simplifies the computation of equilibrium. (Note that the measure consistency criterion is also crucial to pinning down the equilibrium matching pattern.)

9. We thank an anonymous referee for pointing Condition M out to us. Conditions M and H jointly are weaker than a condition discussed in an earlier draft of this paper that also yields median matching.

Second, if the technology or production changes in such a way as to preserve Condition M, the match will be unchanged because there is only one way to have median matching for a given distribution of types.

Third, Condition M (and Condition H) is typically satisfied only on a restricted domain of the production function. Changes to the support of the type distribution may then affect whether or not there is median matching.

*Example.* The post-match task assignment economy conforms to the hypotheses of Proposition 6, provided the support of the type distribution is tight enough. If, for instance,  $h(a, b) = \max\{a^\theta b^{1-\theta}, b^\theta a^{1-\theta}\}$ , Condition M is satisfied on  $[\underline{a}, \bar{a}] \subset R_{++}$  if and only if  $\underline{a} \geq (\frac{1-\theta}{\theta})^{1/\theta} \bar{a}$ . (One can verify that for this model, Condition H holds if M does.)<sup>10</sup>

The main result in Kremer and Maskin (1996) essentially depends on this observation. For a case isomorphic to  $\theta = 2/3$ , they consider changes to a numerical measure of the degree of segregation resulting from a “lengthening” of the support of the type distribution (for their measure, median matching happens to minimize the degree of segregation). When the support of the distribution is small, Condition M is satisfied, median matching results, and the degree of segregation is thereby minimized. As the support is stretched, Condition M will eventually be violated and the matching pattern then becomes more segregated.

One can also derive qualitative results on the effects of technological change in a similar fashion. Fix a continuous distribution of types with compact support and recall that for  $\theta = 1/2$  we have perfect segregation. As  $\theta$  increases to 1 (or decreases to 0), Condition M eventually becomes satisfied and we get median matching, which minimizes the segregation measure.

This suggests that as  $\theta$  varies over the unit interval, there will be a kind of inverted U-shaped relation between the degree of asymmetry of the tasks and the degree of segregation in the economy. In particular, interpreting increases in  $\theta$  beyond 1/2 as a kind of “skill-biased” technical change, one gets from this model the prediction that this kind of technical change should lead to a reduction in the degree of segregation in firms.

#### 4.2. Negative assortative matching

Given the simple logic that led to PAM, sufficient conditions for negative matching are easily identified: one must simply rule out matches that violate negative matching. NAM is a stronger concept than PAM, however (as we have suggested, it is really the counterpart to median matching), and so the conditions guaranteeing it for any type distribution are correspondingly more restrictive

**Proposition 7.** *If the production function is symmetric and strictly submodular on  $A^2$ , the economy is negatively matched.*

*Proof.* Putting  $x = (a, d)$  and  $y = (b, c)$  into the submodularity condition  $h(x) + h(y) > h(x \vee y) + h(x \wedge y)$  yields  $h(a, d) + h(b, c) > h(a, c) + h(b, d)$ ; putting instead  $y = (c, b)$  and using  $h(c, b) = h(b, c)$  yields  $h(a, d) + h(b, c) > h(a, b) + h(c, d)$ . Thus whenever there are four types  $a > b \geq c > d$ , output is higher when they are negatively matched than when

10. To see this, note that  $a^\theta b^{1-\theta} - a/2 - b/2$  is strictly concave in  $b$ , has zeros at  $b = a$  and  $b_0(a, \theta)$ , and is positive on  $(b_0(a, \theta), a)$ . It is straightforward to check that  $b_0(a, \theta)$  is increasing in  $a$ . Similarly,  $a^{1-\theta} b^\theta - a/2 - b/2$  is positive on  $(a, b^0(a, \theta))$  with  $b^0(a, \theta)$  increasing in  $a$ . Thus  $\sigma(\underline{a}, \bar{a}) > 0$  implies that  $\sigma(a, b) > 0$  for  $a, b \in [\underline{a}, \bar{a}], a \neq b$ . Setting  $\underline{a} = (\frac{1-\theta}{\theta})^{1/\theta} \bar{a}$  to ensure Condition M is satisfied,  $\sigma(\underline{a}, \bar{a}) > 0$  is equivalent to  $2(\frac{1-\theta}{\theta})^{\frac{1-\theta}{\theta}} > 1 + (\frac{1-\theta}{\theta})^{1/\theta}$ , which is easily verified.

they are positively matched. Finally, putting  $x = (a, b)$  and  $y = (b, a)$  establishes that  $h(a, b) > \frac{1}{2}[h(a, a) + h(b, b)]$  (and  $h(c, d) > \frac{1}{2}[h(c, c) + h(d, d)]$ ), so  $h(a, d) + h(b, c) > \frac{1}{2}[h(a, a) + h(b, b) + h(c, c) + h(d, d)]$ , which shows that segregation is also dominated. Recalling from Proposition 1 that output is maximized at equilibrium completes the argument. ||

**Definition 7.** The function  $h : A^2 \rightarrow \mathbb{R}$  satisfies weak decreasing differences (WDD) on  $A$  if for any four elements  $a, b, c, d$  of  $A$ , where  $a > b \geq c > d$ ,

$$h(b, c) - h(b, d) \geq h(a, c) - h(a, d)$$

and

$$h(b, c) - h(c, d) \geq h(a, b) - h(a, d).$$

As might be expected by analogy to the case of median matching, this weakening of submodularity (decreasing differences), which for symmetric functions would also entail  $h(a, b) - h(a, c) \leq h(b, d) - h(c, d)$ , along with either continuity or Condition H, is the key condition for NAM.<sup>11</sup>

**Proposition 8.** *The economy is negatively matched if either (1)  $h$  satisfies WDD and is continuous; or (2)  $h$  satisfies WDD and Condition H. If in addition the WDD inequalities hold strictly, then NAM is the unique equilibrium outcome.*

## 5. NECESSARY CONDITIONS AND THE DEPENDENCE OF MONOTONE MATCHING ON DISTRIBUTION

We already mentioned that in a rather weak sense, the match will typically depend on the distribution of types, since unless the economy is always segregated,  $\mathfrak{M}(a)$  will vary as the distribution does. At the same time, such sensitivity to the distribution may be difficult to detect empirically. What is more striking is a loss or reversal of monotonicity when the distribution changes. In order to evaluate when such phenomena are possible, it is useful to have *necessary* conditions for monotonicity independent of the distribution.

We first remark that the sufficient condition for segregation in Proposition 4 is also necessary: if  $\sigma(a, b) > 0$  for some  $a, b \in A$ , then there is a distribution in which matching is not segregated: if half the agents are of type  $a$  and half of type  $b$ , the match will be perfectly heterogeneous. Thus, we have the following proposition.

**Proposition 9.** *A necessary and sufficient condition for the economy to be segregated for all type distributions is that  $\sigma(\cdot, \cdot) \equiv 0$ .*

For PAM and NAM, it turns out that the same monotone difference conditions we saw before play essential roles. However, for the necessary conditions they are applied to the surplus rather than the production function.

**Proposition 10.** *A necessary and sufficient condition for the economy to be positively matched for all type distributions is that for all  $a > b \geq c > d$  with  $\sigma(a, d) > 0$ , the surplus function satisfies the WID inequalities.*

11. It is not hard to find examples of functions which satisfy Condition H and WDD but are not submodular, although it is perhaps not clear how “natural” they are. For instance, let  $a > b > c > d$  and define  $h(\cdot, \cdot)$  to be the symmetric function on  $\{a, b, c, d\} \times \{a, b, c, d\}$  with  $h(a, a) = 80, h(a, b) = 76, h(b, b) = 70, h(a, c) = 62, h(b, c) = 60, h(a, d) = 54, h(c, c) = 44, h(b, d) = 40, h(c, d) = 37, h(d, d) = 8$ . It is straightforward to check that  $h$  satisfies the H and WDD inequalities strictly, but that  $h(a, b) - h(a, c) > h(b, d) - h(c, d)$ , and hence is not submodular.

*Proof.* The proof of sufficiency is straightforward. For necessity, we need to show that if the surplus function does not satisfy the hypotheses of the proposition, there is a type assignment for which the economy is not positively matched. Suppose there exist  $a > b > c > d$  with  $\sigma(a, d) > 0$  such that

$$\begin{aligned} \sigma(a, d) + \sigma(b, c) &> \sigma(a, b) + \sigma(c, d) \\ \text{and} \quad \sigma(a, d) + \sigma(b, c) &> \sigma(a, c) + \sigma(b, d). \end{aligned}$$

Consider an economy with equal masses at the points  $a, b, c, d$ . Since  $\sigma(a, d) > 0$ , segregation cannot be an equilibrium outcome. Neither can the outcomes  $(\{a, b\}, \{c, d\})$  or  $(\{a, c\}, \{b, d\})$ , since they generate less surplus than  $(\{a, d\}, \{b, c\})$ . The remaining possibilities consistent with PAM are

$$\begin{aligned} m_1 &= (\{a, a\}, \{b, b\}, \{c, d\}) \\ m_2 &= (\{a, b\}, \{c, c\}, \{d, d\}) \\ m_3 &= (\{a, a\}, \{b, c\}, \{d, d\}). \end{aligned}$$

Clearly,  $m_1$  and  $m_2$  generate less surplus than  $(\{a, d\}, \{b, c\})$  since they generate no more surplus than  $(\{a, b\}, \{c, d\})$  or  $(\{a, c\}, \{b, d\})$ . Finally,  $m_3$  cannot be an equilibrium since  $\sigma(a, d) > 0$  (note the role of the provision  $\sigma(a, d) > 0$ ). Hence the NAM  $(\{a, d\}, \{b, c\})$ , which is the only other possibility, is the unique equilibrium outcome. If  $b = c$ , the argument is the same. ||

Obviously, if the production function satisfies WID, so does the surplus, but the converse is not true.

*Example.* Suppose that  $A = [0, 1]$  and  $h(a, b) = (a - b)^2$  for  $a \neq b$  and  $2a^2$  for  $a = b$ . The surplus is identically zero (the economy is segregated), and so satisfies WID, but the production function actually satisfies WDD (it is not submodular, however). Nor is it continuous, which shows why the additional hypotheses of continuity or Condition H are needed in Proposition 8.

Thus the economy can be positively matched for all type distributions, even though the production function fails to satisfy WID (we saw this in the first example in Section 4.1 as well). But if the surplus fails to satisfy the WID inequalities at some quadruple  $(a, b, c, d)$  where  $\sigma(a, d) > 0$ , then the economy with atoms of equal measure at those types will be negatively matched.

Similar logic gives the result for negative matching.

**Proposition 11.** *A necessary and sufficient condition for the economy to be negatively matched for all type distributions is that condition H is satisfied and that for all  $a > b \geq c > d$  the surplus function satisfies the WDD inequalities.*

When neither the hypotheses of Proposition 10 (i) nor of Proposition 11 (i) are satisfied, then the matching outcome will be sensitive to the type distribution in a strong sense: it may be positive for some distributions, negative for some, or even non-monotonic for others.

To illustrate the use of these necessary conditions, we turn to the imperfect capital market example.

*Example.* The segregation payoff is

$$\underline{u}(a) = \begin{cases} \frac{a^2 - k}{2} & \text{if } a^2 \geq \phi k \\ 0 & \text{if } a^2 < \phi k. \end{cases}$$

As  $\phi$  increases, so does the set of types with zero segregation payoffs. Divide the type space into two intervals,  $A_0 = [a, \sqrt{\phi k}]$  and  $A_+ = [\sqrt{\phi k}, \bar{a}]$ . For all  $b \in A_0$  the segregation payoff is zero, while for all  $a \in A_+$ , it is the same as if there were no market imperfection. If one takes  $a \in A_+$  and  $b \in A_0$  with  $ab \geq \phi k$  that are not too far apart, the surplus generated is  $ab - \frac{1}{2}(k + a^2) > 0$ . Thus when  $\phi$  is large enough, one can satisfy Condition H for some distributions, and there will be heterogeneous matching.

In fact, the surplus does not satisfy the WID inequalities either: choose  $a > b > d > \frac{1}{2}a + \frac{k}{2a}$  so that  $ad > \phi k > bd$ , and  $b^2 \geq \phi k > d^2$ . Then  $\sigma(a, d) = ad - k - \frac{1}{2}(a^2 - k) > 0$ , while  $\sigma(b, b) = \sigma(b, d) = \sigma(a, b) = 0$ . Thus for this set of types we have  $\sigma(a, d) + \sigma(b, b) > \sigma(a, b) + \sigma(b, d)$  which contradicts WID (clearly, the surplus doesn't satisfy WDD either: where it is differentiable, the cross partial is positive). Proposition 10 then implies that there are type distributions for which the match is not positive assortative.

This example also illustrates a general principle underlying comparative statics of the equilibria of matching models. As environmental (technology, information) parameters change, both utility possibility sets and segregation payoffs change. The way they change may be very different for different types, however. Thus for a model in which the condition for segregation is satisfied at one parameter value, the segregation payoff vector may “move inside” the feasible set of a heterogeneous pair as the parameter increases.

*Example.* If  $a$  is just above  $\sqrt{\phi k}$ , its segregation payoff falls from  $\frac{a^2 - k}{2}$  to zero as  $\phi$  increases, while the same increase in  $\phi$  has no effect on the segregation payoff of types far above  $\sqrt{\phi k}$ . For  $\phi > 1$ , a high type may have an incentive to match not with another high type, but with a low type instead because the latter's outside opportunities are so low. In other words, while the segregation payoff vector must lie outside of a heterogeneous pair's Pareto frontier when  $\phi = 1$ , it may move inside the frontier when  $\phi > 1$ . See Figure 2: for  $\phi = 1$ , the segregation payoff vector  $\underline{u}$  lies outside the utility possibility frontier for a heterogeneous pair  $\{a, b\}$ ; for  $\phi > 1$ , it lies inside the frontier ( $b^2 < \phi k < ab$  is assumed, hence coalition  $\{b, b\}$  cannot borrow while coalitions  $\{a, b\}$  and  $\{a, a\}$  can).

We illustrate the dependence of the matching pattern on the distribution. As we pointed out earlier, despite the financial market imperfection, utility is still transferable within each coalition. Proposition 1 therefore implies that the equilibrium match will maximize aggregate surplus, which will be helpful in computing the equilibrium. Space permits only a sketch of the arguments; see our working paper Legros and Newman (2000) for details.

*Example.* First, note that any heterogeneous match must consist of a type  $a \in A_+$  and a type  $b \in A_0$ . Heterogeneous matches must satisfy two other conditions: first, the financing constraint

$$ab \geq \phi k \tag{3}$$

must be met, and second there must be a gain to matching, which entails

$$ab - k - \frac{1}{2}(a^2 - k) \geq 0. \tag{4}$$

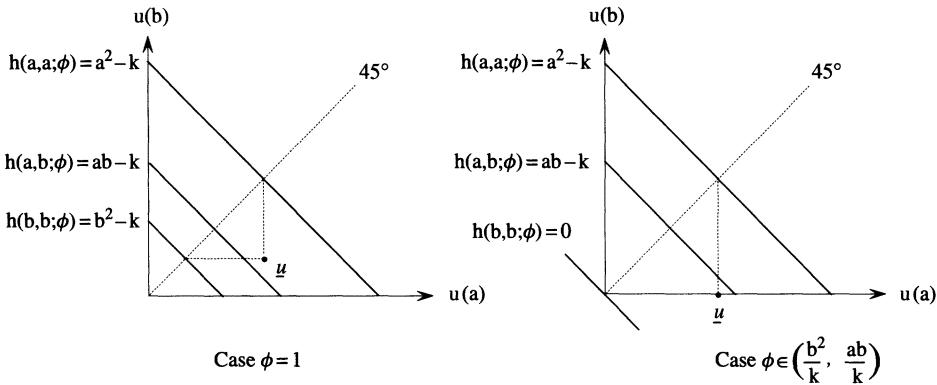


FIGURE 2

Define  $b^*$  to be the lowest value in  $A_0$  for which these constraints can be satisfied by some  $a$  in  $A_+$ ;  $a^*$  as the point in  $A_+$  at which both constraints bind at  $b^*$ ; and  $a^{**}$  the point in  $A_+$  above which (4) cannot be satisfied for any  $b$  in  $A_0$ . We have  $b^* < \sqrt{\phi k} < a^* < a^{**}$ . All agents above  $a^{**}$  segregate, as do any agents below  $b^*$ .

Suppose that the support of the ability distribution is contained in  $[b^*, \sqrt{\phi k}] \cup [a^*, a^{**}]$ . Take  $a > \hat{a}$  in  $[a^*, a^{**}]$  and  $b > \hat{b}$  in  $[b^*, \sqrt{\phi k}]$ . Since  $ab + \hat{a}\hat{b} > \hat{a}\hat{b}$ , the payoff function  $h(a, b, \phi)$  satisfies the WID inequalities, so matching is positive assortative.

Now suppose that the distribution is log uniform on  $[b^*, a^*]$  (that is, the logarithm of ability is uniformly distributed). Consider a relaxed maximization program in which the constraint that the match be measure consistent is ignored. For a given  $b$ , to maximize  $\sigma(a, b) = ab - \frac{1}{2}(a^2 + k)$ , one chooses the smallest element of the set of  $a$  satisfying (3) and (4), which is  $\frac{\phi k}{b}$ . Thus the pointwise optimum has  $\mathfrak{M}(b) = \frac{\phi k}{b}$ , and with log uniformity, this is also measure consistent: the relaxed optimum is the optimum in the original program. Since the larger is  $b$ , the smaller is his partner, we conclude that with this distribution we get NAM.

If the distribution is log uniform on a larger interval, say  $[b^*, a^{**}]$ , we still get NAM on  $[b^*, a^*]$  and segregation elsewhere (more surplus is generated when a  $b$  remains with  $\frac{\phi k}{b}$  than if he switches to an  $a$  in  $[a^*, a^{**}]$ ). This is a non-monotonic matching pattern.

Thus as the distribution changes, so does the matching pattern, in ways that reverse the direction of the monotonicity or destroy it altogether.

**Optimality in the imperfect credit market model.** As we have noted, aggregate output is maximized at the equilibrium of any transferable utility model, in particular the one with imperfect credit markets. The primary impact of changes in the degree of imperfection there is to change the equilibrium matching pattern (of course, total output falls as the imperfection becomes more severe). But the outcome is always optimal from the point of view of maximizing total output.

How do we reconcile optimality of the equilibrium here with the well-known results that say that in the presence of financial market imperfections, equilibrium need not maximize aggregate output? Some models of credit market imperfections introduce non-transferabilities and others generate pecuniary cross-coalitional externalities, both of which are precluded here. Even when it ignores these features, the literature frequently examines mean-preserving redistributions of types (usually interpreted as wealth), from which output increases may be obtained.

By contrast, Proposition 1 merely says that matching will be efficient *given* the distribution of types: there are no output-increasing policies that involve only a reassignment of matches away from those obtained in equilibrium. Indeed, in many transferable-utility matching models, even those not derived from imperfections, one can often raise output if one is allowed to redistribute types. But purely “associational redistribution” (see Durlauf (1997)) can play no efficiency-enhancing role here.

## 6. CONCLUSION

When it comes to determining matching patterns, imperfections can overwhelm technology. A credit market imperfection in an otherwise standard model turns segregation into NAM or something non-monotonic. By the same token, very different economies may generate very similar matches: NAM could signal a perfect market and a submodular technology or an imperfect credit market and a supermodular technology. As evidence of economic processes, matching patterns need to be interpreted with caution.

This point underscores the importance of having conditions for monotone matching for a broader range of environments than those with transferable utility and one-dimensional type spaces. In some cases, this is easy: Proposition 9 and Condition H generalize, essentially via a change of notation, to situations with multiple agents per match, multidimensional characteristics and non-transferability.

For other forms of monotone matching, Legros and Newman (2002a) develop sufficient conditions for the non-transferable utility case in which the utility possibility frontier is strictly decreasing. The same logic inherent in the transferable utility case, namely that (in the case of ID) higher types are always able to outbid lower types in order to match with higher types, leads to the result that “generalized difference conditions” suffice for monotone matching. These have the same formal structure as standard difference conditions, with monotone functions taking the place of real numbers, and functional composition taking the place of real addition. More significantly, there are simple local and supermodular versions of these conditions that facilitate computation and application.

Non-transferable utility also undermines the efficiency of equilibrium matches: a social planner seeking to raise aggregate welfare relative to that of the equilibrium can sometimes do so through associational redistribution, forcing partnerships to form in ways that differ from the equilibrium outcome. Welfare gains can be generated for two distinct reasons: (1) some types become much more productive when matched according to the planner’s preference (inefficiency of the equilibrium match); and (2) in the planner’s preferred matches, partners make more efficient choices (inefficiency by the match). These issues are explored in Legros and Newman (2002b).

The multidimensional case introduces new difficulties (see for instance Engl and Scotchmer (1996)). First of all, there is a myriad of ways in which the various characteristics of the matching partners might enter into their joint output. A most natural way to proceed is to suppose that the characteristics can be summarized by a one-dimensional quantity (call it “talent”). Output then depends on talent in the usual way. Talent is not observable to the investigator, but (some of) the characteristics are (one thinks of athletes, whose height and weight might be easy to measure but whose athletic talent might require the appraisal of experts who match the athletes into teams).

But even in this restricted environment with two-person matches, it is easy to find cases in which matching satisfies PAM in talent but appears as NAM in every dimension observable to the investigator. The problem is that the joint distribution of characteristics leaves a degree of freedom that doesn’t fully nail down the matching pattern. Avoiding this predicament requires a weaker, statistical definition of PAM and related restrictions on the joint distribution of characteristics. We discuss this in Legros and Newman (1999).

## APPENDIX

**Equilibrium.** We provide a definition of equilibrium and of measure consistency in terms of individuals. The economies we study have a continuum of individuals who are designated by the set  $I = [0, 1] \times [0, 1]$  with Lebesgue measure  $\lambda$ . The description of a specific economy includes an assignment of individuals to types via a map  $\tau : I \rightarrow A$ , where unless otherwise specified, the type space  $A$  is taken to be a completely ordered, compact subset of some Euclidean space. The map  $\tau$  is measurable. We also assume that any two individuals with the same first coordinate get assigned the same type by  $\tau$ : if  $i = (x, y)$  and  $j = (x, \hat{y})$ , then  $\tau(i) = \tau(j)$ . The type assignment  $\tau$  induces the distribution of types  $T(\cdot)$  that we use in the text, by the relationship  $T(B) = \lambda(\{i \in I : \tau(i) \in B\})$  for every measurable subset  $B$  of  $A$ .

This construction is appropriate for two reasons. First, the core is defined in terms of individuals rather than types. Secondly, since it is possible that a given type is matched with more than one type, the definition of measure consistency with respect to  $A$  is awkward.

The equilibrium specifies the way individuals are matched to each other, *i.e.* the way the set  $I$  is partitioned into coalitions. Let  $\mathcal{P}$  be a partition of  $I$  into sets of size two. The correspondence  $\mathfrak{M}^* : A \rightrightarrows A$  used in text is defined as

$$\mathfrak{M}^*(a) = \{b : \exists(i, j) \in \mathcal{P} : \tau(i) = a, \tau(j) = b\}.$$

Note that since  $\mathcal{P}$  is a partition,  $\mathfrak{M}^*(a)$  is not empty for all  $a \in A$  and for each  $i \in I$  there exists a unique  $j \in I$  such that  $(i, j) \in \mathcal{P}$ . We say that  $\mathcal{P}$  is measure consistent if for every measurable set  $J \subset I$ , the set of individuals matched with individuals in  $J$  has the same Lebesgue measure as  $J$ :

$$\lambda(J) = \lambda(\{i : \exists j \in J : (i, j) \in \mathcal{P}\}).$$

This restriction rules out partitions in which, for instance, all individuals in  $[0, 1/3] \times [0, 1]$  are matched one-to-one with all the individuals in  $(1/3, 1] \times [0, 1]$  (see also Kaneko and Wooders (1986)).

Denote by  $\mathbb{P}$  the set of *measure consistent* partitions of  $I$  into subsets of size two at most. A partition can be part of an equilibrium if there exists a payoff structure that is feasible for that partition and such that it is not possible for any individuals to obtain a higher payoff by forming a coalition different from their equilibrium coalition.

*Definition A1.* An equilibrium is a pair  $(\mathcal{P}, u)$  consisting of a partition  $\mathcal{P} \in \mathbb{P}$  and a utility allocation  $u : I \rightarrow \mathbb{R}$  such that

- (i)  $u$  is feasible: for almost all  $P = \{i, j\} \in \mathcal{P}$ ,  $(u(i) + u(j)) \leq h(\tau(i), \tau(j))$ .
- (ii)  $u$  cannot be improved upon: there does not exist a pair of individuals  $\{i, j\}$  and payoffs  $(\hat{u}(i), \hat{u}(j))$  such that  $(\hat{u}(i) + \hat{u}(j)) \leq h(\tau(i), \tau(j))$  and  $(\hat{u}(i), \hat{u}(j)) \gg (u(i), u(j))$ .

An equilibrium always exists under our assumptions (see Gretsky *et al.* (1992)).

*Proof of Proposition 2.* Obvious for segregation. For PAM and NAM, the definition requires that the graph of  $\mathfrak{M}$  be monotonic (non-decreasing for PAM, non-increasing for NAM). Consider PAM and assume that there is an open interval  $B \subseteq A$  such that  $\mathfrak{M}(a)$  has more than one element for every  $a \in B$ . Let  $\bar{m}(a) = \sup \mathfrak{M}(a)$  and  $\underline{m}(a) = \inf \mathfrak{M}(a)$ ; by assumption, for each  $a \in B$ ,  $\bar{m}(a) > \underline{m}(a)$ . Now, by PAM,  $\bar{m}$  and  $\underline{m}$  are increasing in  $a$ . Since monotonic functions are continuous almost everywhere, consider a point of continuity  $a$  for  $\bar{m}$  and  $\underline{m}$ . Since  $\bar{m}(a) > \underline{m}(a)$ , by continuity of  $\bar{m}$  and  $\underline{m}$ , there exists  $\varepsilon > 0$  small enough such that  $\bar{m}(a - \varepsilon) > \underline{m}(a + \varepsilon)$ , but this is a contradiction since there is an element of  $\mathfrak{M}(a - \varepsilon)$  that is strictly greater than an element of  $\mathfrak{M}(a + \varepsilon)$ . The proof is similar for NAM.

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