Co-Ranking Mates: Assortative Matching in Marriage Markets

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Abstract

We show that *co-ranking* is the necessary and sufficient condition for assortative matching with strictly nontransferable utility. This condition is equivalent to the GID condition in Legros and Newman (2007) and is a weakening of existing conditions for equilibrium uniqueness.

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1 Introduction

Often in applied work, the investigator is interested in whether the matched partners are sorted according to some characteristics. Sufficient conditions for this outcome to occur in stable matches in two extreme environments of interest were famously provided by Becker (1973) in his pioneering paper on marriage markets. Specifically, in the case of perfectly transferable joint payoffs, as when the joint surplus is a monetary return that can be contractually costlessly divided, increasing differences (ID) of the production function in types is sufficient for positive assortative matching (PAM). In the case of strictly non-transferable utility (strict NTU), as when the produced output is a local public good and commitment to side payments is impossible, Becker also showed that it suffices for PAM that each individual's payoff is *monotonic* in the partner's type.

Recently, Legros and Newman (2007) have considered intermediate environments, where the frontier of the feasible set for a household is strictly decreasing but where the rate of substitution along the frontier is not necessarily equal to -1. They provide a condition on the frontier that leads to PAM; this condition (GID) coincides with Becker's condition when the frontier has slope -1. An important feature of GID, like ID for transferable utility, is that it suffices for PAM regardless of what the relative scarcities of the different characteristics in the economy might be, and is in fact necessary if PAM is to be the outcome for any distribution.

Monotonicity, as it turns out, is not necessary for assortative matching in the strict NTU environment. In this letter we provide a weakening of Becker's condition, called co-ranking, that is necessary as well as sufficient for PAM. We show that this condition coincides with the GID condition in Legros-Newman (2007). Since PAM insures uniqueness of the match, our conditions strengthen those provided by Clark (2006) and Eeckhout (2000) in their studies of the uniqueness of matching outcomes.

2 Assortative Matching in a Marriage Market

The sets of men and women are finite. A woman w has preferences $\succeq_w$ on $M \times M$ and a man of type m has preferences $\succeq_m$ on $W \times W$. A match is a function $\mu : W \cup M \to W \cup M$, where men are assigned to women unless they are left alone (in which case they are matched to themselves).

A match μ is stable if for any $w_i \in W$ and any $m_k \in M$, it is not true that w_i and m_k would strictly prefer to match with each other rather than matching with their partners $\mu(w_i)$ and $\mu(m_k)$, that is either $\mu(w) \succeq_w m$ or $\mu(m)) \succeq_m w$.

We assume that men and women are characterized by their "talent" and that preferences depend only on one's talent and the partner's talent. Let T_W, T_M the sets of possible talent levels for women and men. While the Gale and Shapley marriage market is usually modeled by ordinal preferences, it is convenient to consider a cardinal representation of preferences. Let $f : T_M \times T_W \to \mathbb{R}_+$, $g : T_W \times T_M \to \mathbb{R}_+$ be such that a match between a woman of talent $\tau_w \in T_W$ and a man of talent $\tau_m \in T_M$ generates payoffs $f(\tau_m|\tau_w)$ for the man and $g(\tau_w|\tau_m)$ for the woman. An *economy* consists of the talent sets and the payoff structure (T_W, T_M, f, g) and a *population* (W, M, t), where W and M are the sets of women and men, and where t is the talent assignment with $t(m) \in T_M$ for $m \in M$ and $t(w) \in T_W$ for $w \in W$. Hence,

$$\begin{aligned} m \succeq_w \hat{m} &\quad \Leftrightarrow g\left(t\left(w\right)|t\left(m\right)\right) \geq g\left(t\left(w\right)|t\left(\hat{m}\right)\right) \\ w \succeq_m \hat{w} &\quad \Leftrightarrow f\left(t\left(m\right)|t\left(w\right)\right) \geq f\left(t\left(m\right)|t\left(\hat{w}\right)\right) \end{aligned}.$$

Assortative matching can be defined with respect to a given population.

Definition 1 *Consider an economy with population (W, M, t). An equilibrium match μ satisfies PAM whenever $t(\mu(w)) > t(\mu(\hat{w})) \Rightarrow t(w) \geq t(\hat{w})$ and $t(\mu(m)) \geq t(\mu(\hat{m})) \Rightarrow t(m) \geq t(\hat{m})$.*

Now, not all equilibria for a given population need satisfy PAM. Moreover, even if one could be assured that there was a unique equilibrium for any population, it is perfectly possible for PAM to be satisfied for some populations but not others.

This lack of robustness in the change of the population may be unattractive, and we may want to impose a stronger condition for PAM that is independent of the population.

Definition 2 *An economy satisfies PAM if Definition 1 is satisfied for any population.*

For the purposes of Definition 2, an "economy" is fully specified by the preference ordering of each type over the types on the other side. If an economy does not satisfy Definition 2 then we know that migration, investment in human capital, or other changes in the talent distribution will also affect how agents match qualitatively.

When preferences between partners of different talents can be weak, PAM in the sense of Definition 2 cannot be satisfied: for instance, if w is indifferent between m and m', $m > m'$ while m and m' strictly prefer w to w', matches $(m, w), (m', w')$ constitute an equilibrium but matches $(m, w'), (m', w)$ also constitute an equilibrium. For this reason, we will consider only economies where agents are not indifferent between partners of different talent levels.

Becker (1973) showed that when more talented agents are always preferred as partners, there is PAM; it is also immediate that there is a unique equilibrium. We provide below the necessary and sufficient condition for PAM that is a weakening of Becker's condition.

Co-ranking requires that for any men $m > m'$, and women $w > w'$, either the top man and woman prefer each other to the alternative match or that the bottom man and woman prefer each other to the alternative match.

Definition 3 *Preferences in a population satisfy co-ranking if for any $m > m', w > w'$, either (i) $f(m|w) > f(m|w')$ and $g(w|m) > g(w|m')$ or (ii) $f(m'|w') > f(m'|w)$ and $g(w'|m') > g(w'|m)$.*

Proposition 4 *Co-ranking is necessary and sufficient for PAM.*

Proof. (Necessity) Consider $m > m', w < w'$. Suppose that co-ranking fails; we want to show that the NAM match $\mu(m) = w', \mu(m') = w$ is stable. The coalition (m, w) destabilizes μ only if $f(m|w) > f(m|w')$ and $g(w|m) > g(w|m')$, but then (i) of Definition 3 would be satisfied. The coalition (m', w') destabilizes μ only if $f(m'|w') > f(m'|w)$ and $g(w'|m') > g(w'|m)$, but then (ii) of Definition 3 would be satisfied. Hence, lack of co-ranking allows NAM equilibrium matching.

(Sufficiency) Suppose co-ranking. Consider $m > m', w > w'$, and suppose that $\mu(m) = w'$ and $\mu(m') = w$. Co-ranking, implies that stability is violated for either (m, w) or (m', w'). ■

Co-ranking is a strict weakening of Becker's monotonicity condition. For instance, in the economy with three men and three women and preferences

$$\begin{aligned} m_1 \succ_{w_1} m_3 \succ_{w_1} m_2, w_1 \succ_{m_1} w_3 \succ_{m_1} w_2 \\ m_3 \succ_{w_2} m_1 \succ_{w_2} m_2, w_3 \succ_{m_3} w_1 \succ_{m_2} w_2 \\ m_3 \succ_{w_3} m_2 \succ_{w_3} m_1, w_3 \succ_{m_3} w_2 \succ_{m_3} w_1 \end{aligned}$$

there is co-ranking but Becker's monotonicity condition is violated (and could not be restored by reindexing the agents). In fact what is notable about this example is that in contrast to what is entailed by Becker's condition, not only is there not agreement among the men about the women (and vice versa), but some men (and women) don't have monotonic preferences (both features are also immune to re-ordering of the types).

As a further exploration of the nature of co-ranking, we examine its connection to the GID condition (Legros and Newman 2007) that applies to non-strict forms of NTU in which the frontier is strictly decreasing. One can think of the strict NTU case as the limit of such economies. Indeed, in the general NTU case, the Pareto frontier can be expressed by the function $\phi(m, w, v)$ that defines the maximum payoff to a man m matching with woman w who receives a payoff of v. The *quasi-inverse* $\psi(w, m, u)$ defines the maximum payoff woman w can obtain by matching with a man m who must have a payoff of u.

In the case of strict NTU we have,

$$\phi(m, w, v) = \begin{cases} f(t(m)|t(w)) & \text{if } v \leq g(t(w)|t(m)) \\ 0 & \text{if } v > g(t(w)|t(m) \end{cases} \tag{1}$$

$$\psi(w, m, u) = \begin{cases} g(t(w)|t(m) & \text{if } u \leq f(t(m)|t(w)) \\ 0 & \text{if } u > f(t(m)|t(w)). \end{cases} \tag{2}$$

Note that when $u \leq f(t(m)|t(w))$, $\phi(m, w, \psi(w, m, u)) = u$ while when $u > f(t(m)|t(w))$, $\phi(m, w, \psi(w, m, u)) = f(t(m)|t(w))$. Since talent creates a natural order on men and women, we will abuse notation and ignore the reference to $t(\cdot)$; hence we write $f(m|w)$ instead of $f(t(m|)) t(w))$, etc., keeping in mind that higher values of m (or of w) mean that the talent of m (or of w) is also larger.

Legros and Newman (2007) show that the following condition, is necessary and sufficient for an economy to satisfy PAM in the sense of Definition 2 when $\phi(m, w, v)$ is strictly decreasing on its domain $v \in [0, \psi(w, m, 0)]$.

Definition 5 *ϕ satisfies General Increasing Differences (GID) if for any $m > m', w > w', u \in [0, \psi(w, m', u)]$,*

$$\phi(m, w, \psi(w, m', u)) \geq \phi(m, w', \psi(w', m', u)). \tag{3}$$

Proposition 6 *(Legros and Newman 2007) An economy in which ϕ is strictly decreasing on its domain $v \in [0, \psi(w, m, 0)]$ satisfies PAM if and only if ϕ satisfies GID.*

Proposition 7 *Suppose that agents are not indifferent between partners of different talent, then GID is equivalent to co-ranking.*

Proof. Consider men an women of different talent: $m > m$', and $w > w'$. Suppose that GID holds. Using respectively $u = 0$ and $u = f(m'|w)$ in (3), we have

$$\begin{aligned} \phi(m, w, g(w|m')) &\geq \phi(m, w', g(w'|m')) && (4) \\ \phi(m, w, g(w|m')) &\geq \phi(m, w', \psi(w', m', f(m'|w))). && (5) \end{aligned}$$

Suppose first that $g(w|m') > g(w|m)$. Then the left hand side of (4) and of (5) is equal to zero. For (4) we must have $\phi(m, w', g(w'|m')) = 0$, or $g(w'|m') > g(w'|m)$. For (5) we must have $\psi(w', m', f(m'|w)) > g(w'|m)$, which requires that $f(m'|w) < f(m'|w')$. Therefore we have (ii) in Definition 3.
Suppose now that $g(w|m') < g(w|m)$. Then, the right hand side of (4) and (5) is equal to $f(m|w)$. If $f(m|w) > f(m|w')$ we have (i) of Definition 3 and the result is established. Suppose therefore that $f(m|w) < f(m|w')$. It follows that the right hand sides of (4) and (5) cannot be equal to $f(m|w')$, and therefore we need for (5) $\psi(w', m', f(m'|w)) > g(w'|m)$: this requires that $f(m'|w) < f(m'|w')$ – for otherwise $\psi(w', m', f(m'|w)) = 0$ – and $g(w'|m') > g(w'|m)$; but then (ii) of Definition 3 is satisfied. Hence GID implies co-ranking.
We prove now that co-ranking implies GID. Assume first that (i) of Definition 3 holds: $f(m|w) > f(m|w')$ and $g(w|m) > g(w|m')$. Then, for $u \in [0, f(m'|w)]$, the definition of ϕ and ψ imply

$$\begin{aligned} \phi(m, w, \psi(w, m', u)) &\geq \phi(m, w, g(w|m')) \\ &= f(m|w) \\ &> f(m|w') \\ &\geq \phi(m, w', \psi(w', m', u)), \end{aligned}$$

proving GID. Suppose that part (ii) of Definition 3 holds: $f(m'|w') > f(m'|w)$ and $g(w'|m') > g(w'|m)$. Then, for $u \in [0, f(m'|w)]$,

$$\begin{aligned} \phi(m, w', \psi(w', m', u)) &\leq \phi(m, w', \psi(w', m', f(m'|w))) \\ &= \phi(m, w', g(w'|m')) \\ &= 0, \end{aligned}$$

and GID holds. This proves the result. ■

3 Equilibrium Uniqueness

Under the conditions of Corollary 7 there exists a unique equilibrium match between agents. This result is also a direct consequence of a previous result by Clark (2006) who shows that the following "no-crossing" condition is sufficient for uniqueness of the equilibrium when no man is indifferent between two women and no woman is indifferent between two men.[1]

Proposition 8 *(Clark 2006) Consider a population that can be ordered in such a way that if* $w' < w$ *and*

[1] Eeckhout (2000) gives a weaker condition than Clark (2006) for uniqueness of equilibrium *in a given population.* However the condition in Eeckhout fails to hold if for instance some agents leave the population.

$m' < m$,
(Ci) it is not the case that $[g(w'|m) > g(w'|m')$ and $g(w|m') > g(w|m)]$
(Cii) it is not the case that $[f(m'|w) > f(m'|w')$ and $f(m|w') > f(m|w)]$.
Then there exists a unique equilibrium match.

GID implies (Ci)-(Cii). If (i) in Definition 3 holds, $f(m|w) > f(m|w')$ implies (cii) and $g(w|m) > g(w|m')$ implies (Ci). On the other hand it is not hard to generate examples in which (Ci)-(Cii) holds but GID fails.

The initial population has 3 men and women and the ordering is $i = 1, 2, 3$ for men and women. Preferences are strict and are given by

$$\begin{array}{ll} w_1 \succ_{m_1} w_2 \succ_{m_1} w_3 & m_1 \succ_{w_1} m_2 \succ_{w_1} m_3 \\ w_2 \succ_{m_2} w_1 \succ_{m_2} w_3 & m_2 \succ_{w_2} m_3 \succ_{w_2} m_1 \\ w_3 \succ_{m_3} w_2 \succ_{m_3} w_1 & m_3 \succ_{w_3} m_2 \succ_{w_3} m_1 \end{array}$$

Consider a 4-tuple m_k, m_l, w_i, w_j we cannot have a violation of (Ci) or of (Cii). This is immediate if $k = i = 1$ or $l = j = 3$ since in this case the man and the woman of same index prefer to be matched. When $l = 2$ and $i = 2$, since $w_2 \succ_{m_1} w_3$, while $w_2 \succ_{m_2} w_1$, (Cii) holds, and since $m_2 \succ_{w_3} m_1$ while $m_2 \succ_{w_2} m_1$, (Ci) holds. If $k = 2$ and $j = 2$, then $m_2 \succ_{w_1} m_3$ and $m_2 \succ_{w_2} m_3$ imply that (Ci) holds; $w_2 \succ_{m_3} w_1$ and $w_2 \succ_{m_2} w_1$ imply that (Cii) holds. Hence, conditions (Ci)-(Cii) are verified and there is a unique equilibrium but this equilibrium is not PAM necessarily.

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